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MAY 81 P MARKOWICH, M RENARDY DAAG29-80-C-0881

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DESCRIBING THE STRETCHING  
OF POLYMERIC LIQUIDS.

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A NONLINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATION  
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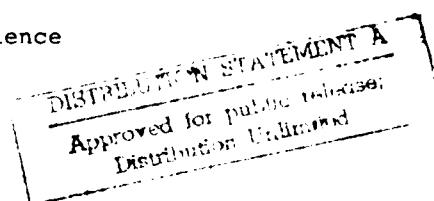
ABSTRACT

We study a model equation for the elongation of filaments or sheets of polymeric liquids under the influence of a force applied to the ends. Mathematically this equation has the form of a nonlinear Volterra integro-differential equation with the kernel given by a finite sum of exponentials. The unknown function denotes the length of the filament or, respectively, the thickness of the sheet. We study the equation both analytically and numerically. The force is assumed to converge to zero exponentially as  $t \rightarrow -\infty$  and to vanish identically after a finite time  $t_0$ . It is shown that under this condition there is a unique solution which approaches a given limit as  $t \rightarrow -\infty$ ; moreover, the solution also has a limit as  $t \rightarrow +\infty$ . A numerical scheme is analyzed and convergence uniformly in  $t$  is established. Particular attention is paid to the dependence of solutions on a parameter  $\mu$ , which corresponds to a Newtonian contribution to the viscosity. It is proved that solutions converge uniformly in  $t$  as  $\mu \rightarrow 0$ , and that the convergence of the numerical scheme is also uniform in  $\mu$ .

AMS (MOS) Subject Classifications: 34D05, 34D15, 45J05, 45L10, 65R20, 76A10

Key Words: Viscoelastic Liquids, Nonlinear Volterra Integrodifferential Equations, Singular Perturbation, Numerical Approximation on Infinite Intervals

Work Unit Number 3 - Numerical Analysis and Computer Science



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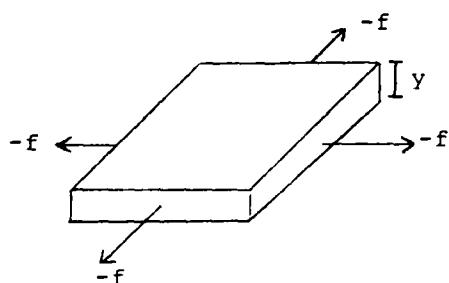
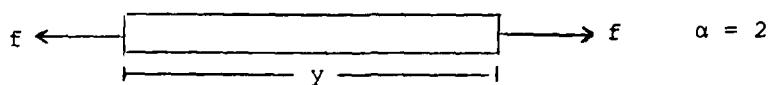
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### SIGNIFICANCE AND EXPLANATION

The evolution of the shape of a filament or a sheet of a polymeric liquid subjected to an external force  $f(t)$  is described by the equation

$$\mu \dot{y}(t) + \int_{-\infty}^t a(t-s) \left( \frac{y^3(t)}{y^2(s)} - y(s) \right) ds = f(t) y^\alpha(t)$$

where  $y$  denotes the length of the filament or the thickness of the sheet, respectively, and  $\mu$  is a Newtonian contribution to the viscosity which can either be positive or zero. The exponent  $\alpha$  depends on the physical situation under study.



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We regard the length at  $t = -\infty$  as known and investigate its evolution. It is shown that for a physically realistic class of functions  $f$  there exists a unique solution, and that the length approaches a new stationary value at  $t = \infty$  (which is in general different (greater) from the value at  $t = -\infty$ ). Numerical calculations are performed for several functions  $f$ . For the kernel  $a$  we choose values given in the literature for polyethylene at 150°C. Our computations show in particular that the solutions do not significantly depend on  $\mu$  unless  $\mu$  exceeds  $10\,000 \text{ Nm}^{-2}\text{sec}$ .

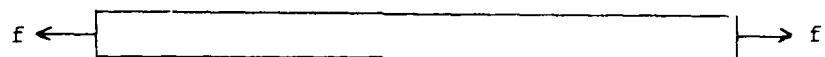
A NONLINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATION  
DESCRIBING THE STRETCHING OF POLYMERIC LIQUIDS

P. Markowich\* and M. Renardy\*\*

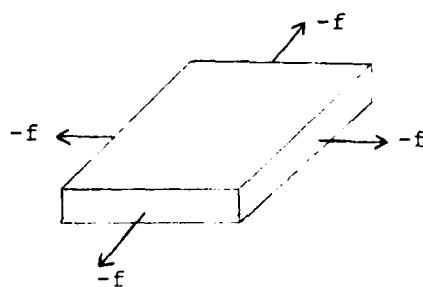
1. Introduction

In this paper we consider a mathematical model describing the stretching of a filament or a sheet of a molten polymer under a prescribed force  $f$ . These two physical situations are illustrated by the following diagrams:

(1)



(2)



Our model is based on the following physical assumptions:

- (i) The polymer satisfies the "rubberlike liquid" constitutive relation [5].
- (ii) The strain and stress tensors are independent of spatial coordinates, and, in particular, inertial forces are neglected (for a model that includes inertial forces see [9]).
- (iii) The molten polymer is incompressible.

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Under these assumptions the problem is described by the equation (for a derivation see [6], [9]):

$$(1.1) \quad \mu \dot{y}(t) + \int_{-\infty}^t a(t-s) \left( \frac{y^3(t)}{y^2(s)} - y(s) \right) ds = f(t) y^\alpha(t), \quad -\infty < t < \infty$$

where  $y$  denotes the length of the filament or the thickness of the sheet, resp.,  $\mu$  is a nonnegative material constant modelling a Newtonian contribution to the viscosity, which may physically come from a solvent or fractions of low molecular weight, and the memory kernel  $a$  has the form

$$(1.2) \quad a(u) = \sum_{\ell=1}^N K_\ell e^{-\lambda_\ell u}$$

with positive constants  $K_\ell$ ,  $\lambda_\ell$ .  $f$  denotes the force acting on the ends of the filament, or  $-f$  denotes the force acting on the edges of the sheet, resp. The exponent  $\alpha$  is 2 for the filament and  $\frac{1}{2}$  for the sheet (for our mathematical analysis, we assume  $0 < \alpha < 3$ ). The difference comes from geometric reasons: If  $f$  were denoting the force per unit area,  $\alpha$  would be 1 for both cases. Due to the incompressibility, however, the area on which  $f$  is acting depends on  $y$ .

Although this has no significance to the mathematical analysis, the physical relevance of the model is limited to  $f > 0$  for the filament and  $f < 0$  for the sheet. If e.g. one attempts to compress the filament, then buckling rather than contraction would be observed, and this instability is not described by our equation.

A problem related to ours was investigated by Lodge, McLeod and Nohel [6]. They assume  $y(t)$  is given for  $t < 0$ , it is nondecreasing (which implies but does not follow from  $f > 0$ ), and  $y(-\infty) = 1$ . They then assume

$f = 0$  for  $t > 0$  and study the elastic recovery. For a class of kernels  $a$  and functions  $F(y(t), y(s))$  under the integral, which include those specified above, they prove the existence of a unique solution to the history value problem, which is nondecreasing for  $t > 0$  and converges to a limit  $y(\infty) > 1$ . Their proofs rely on monotonicity arguments, and they also prove that the solutions depend monotonically on the prescribed history and the parameter  $\mu$ . One of the main points in their analysis is the behavior of solutions near  $\mu = 0$ , in this case the solutions become discontinuous at  $t = 0$ , and they face a singular perturbation problem with a boundary layer. On the basis of these results Nevanlinna [8] used an implicit first order Euler-type discretization scheme for (1.1). He proved that this discretization preserves all the monotonicity properties, and that the global error is  $O(h^\gamma)$  for any  $\gamma < 1$  uniformly with respect to  $\mu \in [0, \mu]$  and  $t \in [t_0, \infty)$ ,  $t_0 > 0$ . It was not shown, however, that the scheme is first order accurate uniformly in  $t$  and  $\mu$ , i.e. that the global error is  $O(h)$ .

In our analysis, we prescribe a continuous function  $f(t)$ , which satisfies  $\lim_{t \rightarrow \infty} e^{-\sigma t} f(t) = 0$  for some  $\sigma > 0$ , and  $f \geq 0$  for  $t \in [t_0, \infty)$ . We prove that, for any such  $f$ , problem (1.1) has a unique solution  $y(t)$  satisfying  $\lim_{t \rightarrow \infty} y(t) = 1$ . This convergence is exponential, moreover, the solution exists globally in time, and converges exponentially to a constant  $y(\infty) > 0$  as  $t \rightarrow \infty$ , more precisely, we have  $\lim_{t \rightarrow \infty} e^{-\sigma t} (y(t) - 1) = \lim_{t \rightarrow \infty} e^{\sigma t} (y(t) - y(\infty)) = 0$ . This holds for any  $\mu > 0$ . The solution depends continuously on  $\mu$  in a norm stronger than the  $L^\infty$ -norm (more specifically,

in an exponentially weighted  $L^\infty$ -norm, which incorporates the asymptotic behavior as  $t \rightarrow \pm\infty$ , even at  $\mu = 0$ . No boundary layer occurs, since the solution for  $\mu = 0$  has the correct asymptotic behavior as  $t \rightarrow \pm\infty$ . Our proofs are mainly based on the implicit function theorem and Liapunov function arguments.

In the second part of the paper we discuss the computational solution of (1.1). Like Nevanlinna, we use a first order implicit Euler-type discretization with uniform mesh size  $h$ , after having cut the interval  $[-\infty, 0]$  at  $t_m$ . In the convergence proof, we use a discrete analogue of exponentially weighted  $L^\infty$ -spaces (infinite sequences converging exponentially on both sides). Choosing a space with an exponential weight given by  $e^{(\sigma-\varepsilon)|t|}$ ,  $0 < \varepsilon < \sigma$ , we obtain an error estimate of the form  $O(h)^{-\varepsilon|t_m|} + o(e^{-\varepsilon|t_m|})$  in the norm of that space, moreover, this holds uniformly in  $\mu \in [0, \infty)$  and  $\varepsilon \in [0, \varepsilon_0]$ ,  $\varepsilon_0 < \sigma$ . The main tool in the proof is Keller's [3] nonlinear stability concept.

Our numerical results imply that the solution  $y(t, \mu)$  does not differ significantly from  $y(t, 0)$  on  $[-\infty, \infty]$  if  $\mu$  is smaller than a certain fairly large number. If  $\mu$  exceeds this number, then the solutions change considerably.

The paper is organized as follows: In chapter 2 we present the analytical results, chapter 3 concerns the discretization procedure, and the computations are reported in chapter 4.

2. Analysis of the Continuous Problem

Solutions for Small Forces

Let us consider equation (1.1), where  $0 < \alpha < 3$ ,  $a(u) = \sum_{i=1}^N K_i e^{-\lambda_i u}$ ,

and  $\mu > 0$ . This equation can be reduced to a system of ODE's in two ways.

We set

$$g_i(t) = \int_{-\infty}^t K_i e^{-\lambda_i(t-s)} \frac{1}{y^2(s)} ds$$

$$h_i(t) = \int_{-\infty}^t K_i e^{-\lambda_i(t-s)} y(s) ds .$$

Then (1.1) reads

$$\dot{y} = -\frac{1}{\mu} \left( \sum_{i=1}^N (g_i y^3 - h_i) - f(t) y^\alpha \right)$$

$$(2.1) \quad \dot{g}_i = -\lambda_i g_i + \frac{K_i}{y^2}$$

$$\dot{h}_i = -\lambda_i h_i + K_i y .$$

If we set  $\gamma_i = g_i y^2$ ,  $\delta_i = \frac{h_i}{y}$ , we obtain

$$\dot{y} = -\frac{1}{\mu} \left( \sum_{i=1}^N (\gamma_i - \delta_i) y - f(t) y^\alpha \right)$$

$$(2.2) \quad \dot{\gamma}_i = -\lambda_i \gamma_i + K_i - \frac{2}{\mu} \gamma_i \sum_j (\gamma_j - \delta_j) + \frac{2}{\mu} \gamma_i f(t) y^{\alpha-1}$$

$$\dot{\delta}_i = -\lambda_i \delta_i + K_i + \frac{1}{\mu} \delta_i \sum_j (\gamma_j - \delta_j) - \frac{1}{\mu} \delta_i f(t) y^{\alpha-1} .$$

Both forms (2.1) and (2.2) will be used in the following.

Clearly, if  $f = 0$ , then  $y = 1$ ,  $g_i = h_i = \frac{K_i}{\lambda_i} = \gamma_i = \delta_i$  is a stationary solution.

LEMMA 2.1. The  $2N + 1$ -square matrix setting up the right hand side of the linearization of (2.1) (or (2.2)) at the stationary solution  $y = 1$ ,  $g_i = h_i = \frac{K_i}{\lambda_i}$  has zero as a simple eigenvalue. All other eigenvalues have negative real parts.

PROOF: Clearly, (2.1) and (2.2) give the same eigenvalues. Let us consider (2.1). The linearization is set up by the following matrix

$$A = \begin{bmatrix} -\sum_{i=1}^N \frac{3K_i}{\mu\lambda_i} & -\frac{1}{\mu} & -\frac{1}{\mu} & \dots & -\frac{1}{\mu} & \frac{1}{\mu} & \frac{1}{\mu} & \dots & \frac{1}{\mu} \\ -2K_1 & -\lambda_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ -2K_2 & 0 & -\lambda_2 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -2K_N & 0 & 0 & \dots & -\lambda_N & 0 & 0 & \dots & 0 \\ K_1 & 0 & 0 & \dots & 0 & -\lambda_1 & \dots & 0 & \\ K_2 & 0 & 0 & \dots & 0 & 0 & -\lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ K_N & 0 & 0 & \dots & 0 & 0 & 0 & \dots & -\lambda_N \end{bmatrix}$$

This yields the characteristic polynomial

$$P(\lambda) = \prod_j (-\lambda_j - \lambda)^2 \left( -\sum_i \frac{3K_i}{\mu\lambda_i} - \lambda - \sum_i \frac{3K_i}{\mu(-\lambda_i - \lambda)} \right) .$$

Thus  $N$  eigenvalues are given by  $\lambda = -\lambda_i$ , the remaining  $N + 1$  eigenvalues are the zeros of the last factor. Obviously one of these is zero, and it is simple. It remains to be proved that all the remaining roots have negative real parts. Consider the equation

$$(2.3) \quad -\sum_i \frac{3K_i}{\mu\lambda_i} - \lambda - \sum_i \frac{3K_i}{\mu(-\lambda_i - \lambda)} = 0 .$$

The left hand side has poles at  $\lambda = -\lambda_i$ , and its sign is positive for  $\lambda \rightarrow -\lambda_i^+$  and negative for  $\lambda \rightarrow -\lambda_i^-$ . For convenience, let the  $\lambda_i$ 's be ordered such that  $\lambda_1 < \lambda_2 < \dots < \lambda_N$ . It follows that there is a root in each interval  $(-\lambda_i, -\lambda_{i+1})$  and another root between  $-\lambda_N$  and  $-\infty$ . Hence all non-zero roots are real and negative.

We want to prove the existence of solutions for small  $f$  using the implicit function theorem. The spaces in which we apply this theorem are defined in the following:

DEFINITION 2.2. Let  $Y^{\sigma, n} = \{g \in C^n(\mathbb{R}, \mathbb{R}) \mid \lim_{|t| \rightarrow \infty} e^{\sigma|t|} g^{(k)}(t) = 0 \text{ for } k = 0, 1, \dots, n\}$ . A natural norm in  $Y^{\sigma, n}$  is

$$\|g\| = \sum_{k=0}^n \sup_{t \in \mathbb{R}} |e^{\sigma|t|} g^{(k)}(t)| .$$

Moreover, let  $X^{\sigma, n} = \{f \in C^n(\mathbb{R}, \mathbb{R}) \mid \lim_{|t| \rightarrow \infty} e^{\sigma|t|} f^{(k)}(t) = 0 \text{ for } k = 1, \dots, n\}$ ,

$$\exists f(\infty) \text{ such that } \lim_{t \rightarrow \infty} e^{\sigma t} (f(t) - f(\infty)) = \lim_{t \rightarrow \infty} e^{-\sigma t} f(t) = 0 .$$

A natural norm in  $X^{\sigma, n}$  is

$$\|f\| = \sum_{k=1}^n \sup_{t \in \mathbb{R}} |e^{\sigma|t|} f^{(k)}(t)| + \sup_{t < 0} |e^{-\sigma t} f(t)| + \sup_{t > 0} |e^{\sigma t} (f(t) - f(\infty))| + |f(\infty)| .$$

THEOREM 2.3. Let  $Y$  denote  $(y, y_1, y_2, \dots, y_N, \delta_1, \delta_2, \dots, \delta_N)$  and  $y_0 = (1, \frac{K_1}{\lambda_1}, \dots, \frac{K_N}{\lambda_N}, \frac{K_1}{\lambda_1}, \dots, \frac{K_N}{\lambda_N})$ . Let  $\sigma > 0$  be small enough (smaller than all the absolute values of the non-zero eigenvalues of  $A$ ). Then the following holds: If  $f \in Y^{\sigma, n}$  has sufficient small norm, then (2.2) has a solution  $y$  satisfying  $y - y_0 \in X^{\sigma, n+1} \times (Y^{\sigma, n+1})^{2N}$ .  $y$  depends smoothly on  $f$ .

Proof: When we put  $Y - Y_0 = Z$ , equation (2.2) can be written in the form

$G(Z, f) = 0$ , and  $G$  is a smooth mapping from  $(X^{\sigma, n+1} \times (Y^{\sigma, n+1})^{2N}) \times Y^{\sigma, n}$  into  $(Y^{\sigma, n})^{2N+1}$ . Moreover, the linearization  $D_Z G(0,0)$  is the mapping

$(y, \gamma_i, \delta_i) \mapsto (-\mu \dot{y} + \sum_{i=1}^N (\gamma_i - \delta_i), \dot{\gamma}_i + \lambda_i \gamma_i + \frac{2K_i}{\mu \lambda_i} \sum_j (\gamma_j - \delta_j), \dot{\delta}_i + \lambda_i \delta_i - \frac{K_i}{\mu \lambda_i} \sum_j (\gamma_j - \delta_j))$ . According to lemma 2.2, the  $\gamma$  and  $\delta$  components form an isomorphism from  $(Y^{\sigma, n+1})^{2N}$  onto  $(Y^{\sigma, n})^{2N}$ . Moreover, the mapping  $y \mapsto \dot{y}$  is a bijection from  $X^{\sigma, n+1}$  on  $Y^{\sigma, n}$ . Therefore  $D_Z G(0,0)$  is an isomorphism from  $X^{\sigma, n+1} \times (Y^{\sigma, n})^{2N}$  onto  $(Y^{\sigma, n})^{2N+1}$ . The implicit function theorem yields the result.

### Global Behavior of Solutions for Large f

Theorem 2.4. Let  $\mu > 0$  and  $f: \mathbb{R} \rightarrow \mathbb{R}$  be continuous and such that

$$\lim_{t \rightarrow \infty} e^{-\sigma t} f(t) = 0 \quad (\sigma > 0 \text{ is as in Theorem 2.3}), \quad f(t) = 0 \text{ for } t > t_0. \quad \text{For}$$

every such  $f$ , equation (2.2) has a unique solution satisfying  $\lim_{t \rightarrow \infty} y(t) = 1$ ,

$$\lim_{t \rightarrow \infty} y_i = \lim_{t \rightarrow \infty} \delta_i = \frac{k_i}{\lambda_i}. \quad \text{This solution exists globally for all times } t,$$

and  $\lim_{t \rightarrow \infty} y(t)$  exists and is strictly positive.

Proof: If  $t_1$  is chosen large enough,  $e^{-\sigma t} f(t)$  becomes small on  $(-\infty, -t_1)$  and one can use an implicit function argument analogous to theorem 2.3 to prove the existence of a solution on  $(-\infty, -t_1)$ . This solution is unique in the class of solutions approaching their limiting values at  $t = -\infty$  at a rate of  $e^{\sigma t}$ . However, if a solution tends to these limits at all, it can be seen from the last two equations of (2.2) and the implicit function theorem that  $y_i$  and  $\delta_i$  tend to their limiting values at a rate of  $e^{\sigma t}$ . The first equation then implies that  $y$  approaches its limiting value at the same rate. Hence the solution is actually unique in the class of all solutions approaching the prescribed limits at  $t = -\infty$  as claimed in the theorem.

We now continue this solution to the right, and we have to make sure that it does not blow up at a finite time. For that purpose it is more convenient to consider (2.1) rather than (2.2). From the second and third equation we see that as long as  $y$  stays positive,  $q_i$  and  $h_i$  have a positive lower bound for all finite times, which is independent of  $y$ . Hence, if  $y$  becomes too large,  $q_i y^3$  will dominate over  $fy^\alpha$  and also over  $h_i$  (since this is less than some constant times  $\max_{(-\infty, t]} y(\tau)$ ). Analogously, if  $y$  becomes too small,  $h_i$  will dominate over  $fy^\alpha$  and  $q_i y^3$ . Hence  $y$  cannot go to zero or infinity in finite time, whence we find global existence.

Let now  $t > t_0$ . Then  $f = 0$ , and using (2.2) again, we find the Liapunov function

$$(2.4) \quad \begin{aligned} & \sum_{i=1}^N \left[ \frac{\mu}{2} \frac{\alpha_i \dot{\alpha}_i}{\alpha_i + \frac{K_i}{\lambda_i}} + \mu \frac{\beta_i \dot{\beta}_i}{\beta_i + \frac{K_i}{\lambda_i}} \right] \\ & = - \sum_{i=1}^N \left[ \frac{\lambda_i \mu}{2} \frac{\alpha_i^2}{\alpha_i + \frac{K_i}{\lambda_i}} + \lambda_i \mu \frac{\beta_i^2}{\beta_i + \frac{K_i}{\lambda_i}} \right] - \left[ \sum_{i=1}^N (\alpha_i - \beta_i) \right]^2. \end{aligned}$$

Here we have put  $\alpha_i = \gamma_i - \frac{K_i}{\lambda_i}$ ,  $\beta_i = \delta_i - \frac{K_i}{\lambda_i}$ . As we know that  $\gamma_i$  and  $\delta_i$  stay positive, the denominators  $\alpha_i + \frac{K_i}{\lambda_i}$ ,  $\beta_i + \frac{K_i}{\lambda_i}$  are always positive, and the left side of the equation (2.4) is thus the derivative of a positive definite function that decreases along trajectories. As an immediate consequence we obtain that  $\alpha_i$  and  $\beta_i$  tend to 0 exponentially for  $t \rightarrow \infty$ . One easily concludes from (2.2) that  $\ln y$  approaches a constant, and hence  $\lim_{t \rightarrow \infty} y(t) > 0$  exists.

The next corollary provides information on the final recovery for physically significant forces.

Corollary 2.5. If  $f$  is always non-negative and not identically zero, then  $y(\infty) > y(-\infty)$ , if  $f$  is always non-positive and not identically zero, then  $y(\infty) < y(-\infty)$ .

Proof: Assume  $f \geq 0$ , the other case is analogous. It is immediate from the integral equation (1.1) that  $f \geq 0$  implies  $y \geq 1$  for all  $t$ . Moreover, if  $f \neq 0$ , there must be some  $t^*$  such that  $y(t^*) > 1$ . Let now  $z(t) = \min_{\tau \in [t^*, t]} y(\tau)$ . Then (1.1) implies that

$$[\frac{d}{dt}]_+ z(t) \geq \min(0, -\frac{1}{\mu} \int_{-\infty}^t a(t-s)(z^3(t)-1)ds)$$

$$\geq -\frac{1}{\mu} \int_{-\infty}^t a(t-s)(z^3(t)-1)ds .$$

If  $z(t) - 1$  is sufficiently small, this gives an inequality of the form

$$[\frac{d}{dt}]_+ z(t) \geq -Ce^{-kt}(z-1) .$$

It follows immediately that  $\lim_{t \rightarrow \infty} z(t) > 1$ .

Remark: These results are obviously expected on a physical basis. Namely, they simply state that pulling the filament effectively increases its length ( $f > 0$ ,  $\alpha = 2$ ) or the thickness of the sheet decreases ( $f < 0$ ,  $\alpha = \frac{1}{2}$ ), resp.

We now give an argument showing that theorem 2.4 does not hold, if the condition that  $f(t) = 0$  for  $t > t_0$  is replaced by exponential decrease of  $f$  and  $\alpha \neq 1$  (in case  $\alpha = 1$  the previous argument still goes through, the only difference being that  $f(t) \sum_{i=1}^N (\alpha_i - \beta_i)$  has to be added on the right side of (3.1)). We restrict ourselves to the case  $N = 1$ . (2.1) reads:

$$-\mu \dot{y} = gy^3 - h - f(t)y^\alpha$$

$$\dot{g} = -\lambda g + \frac{K}{y^2}$$

$$\dot{h} = -\lambda h + Ky .$$

We solve these equations for  $t > 0$  by the following ansatz:

$$y = y_0 e^{vt}, g = g_0 e^{-2vt} + g_1 e^{-\lambda t}, h = h_0 e^{vt} + h_1 e^{-\lambda t} ,$$

$$f = f_0 e^{(1-\alpha)vt} + g_1 y_0^{3-\alpha} e^{((3-\alpha)v-\lambda)t} - h_1 y_0^{-\alpha} e^{(-\alpha v-\lambda)t} .$$

After some calculation one finds that this satisfies the equations if

$$g_0 = \frac{K}{y_0^2(\lambda-2v)}, \quad h_0 = \frac{Ky_0}{\lambda+v} \quad \text{and}$$
$$f_0 y_0^{\alpha-1} = \frac{3vK + \mu v(\lambda-2v)(\lambda+v)}{(\lambda-2v)(\lambda+v)} \quad .$$

We thus find solutions where  $f$  goes to zero exponentially, but  $y \rightarrow \infty$  for  $\alpha > 1$  and  $y \rightarrow 0$  for  $\alpha < 1$ .

All we have to make sure is that by appropriate continuation for  $t < 0$  we can match the conditions at  $t = -\infty$ . For this purpose continue  $y$  in an arbitrary way to the left such that  $y$  is smooth and approaches 1 exponentially at  $t = -\infty$ . The equations for  $g$  and  $h$  then have unique solutions approaching  $\frac{K}{\lambda}$  for  $t \rightarrow -\infty$ . These solutions can be matched to the solutions for  $t > 0$  by appropriate choice of  $g_1$  and  $h_1$ . Finally  $f$  is determined by the first equation.

The Case  $\mu = 0$

In this case the first equation of (2.1) becomes

$$y^3 \cdot \sum_{i=1}^N g_i - \sum_{i=1}^N h_i - f(t)y^\alpha = 0 .$$

Proposition 2.6. For any  $g > 0$ ,  $h > 0$  and  $0 < \alpha < 3$  the equation  $F(y) = gy^3 - h - fy^\alpha = 0$  has a unique solution in  $(0, \infty)$ .

Proof: We have  $F(0) < 0$ ,  $\lim_{y \rightarrow \infty} F(y) > 0$ , so there is clearly a positive

solution. To show it is unique, we investigate zeros of  $F'(y)$ . We have  $F'(y) = 3gy^2 - \alpha fy^{\alpha-1}$ . If  $y > 0$  and  $F'(y) = 0$ , we find  $F(y) = \frac{1}{\alpha} yF'(y) + y^3(1 - \frac{3}{\alpha})g - h < 0$ . This means  $F$  cannot have a positive maximum, whence the result.

The solution  $y(g, h, f)$  can then be inserted into the other equations, yielding a system of  $2N$  equations.

Theorem 2.7. The same statement as in Theorem 2.4 holds also for  $\mu = 0$ .

Also, Corollary 2.5 still holds.

Sketch of the Proof: The existence of a solution on  $(-\infty, -t_1)$  and global existence in time are proved in the same manner as before, and we do not repeat the arguments. If  $f = 0$ , one finds from (2.2)

$$\begin{aligned}\dot{\gamma}_i &= -\lambda_i \gamma_i + \kappa_i + 2\gamma_i \frac{\dot{y}}{y} \\ \dot{\delta}_i &= -\lambda_i \delta_i + \kappa_i - \delta_i \frac{\dot{y}}{y} .\end{aligned}$$

This leads to

$$\sum_{i=1}^N \left[ \frac{1}{2} \frac{\dot{\alpha}_i \alpha_i}{\kappa_i} + \frac{\dot{\beta}_i \beta_i}{\kappa_i} \right] = - \sum_{i=1}^N \left[ \frac{\lambda_i \alpha_i^2}{2(\alpha_i + \frac{\lambda_i}{\kappa_i})} + \frac{\lambda_i \beta_i^2}{\beta_i + \frac{\lambda_i}{\kappa_i}} \right] + \frac{\dot{y}}{y} \cdot \sum_{i=1}^N (\alpha_i - \beta_i)$$

where  $\alpha_i$  and  $\beta_i$  are defined as before.

Since  $\sum_i (\alpha_i - \beta_i)$  is now equal to zero, we still find that  $\alpha_i$  and  $\beta_i$  approach 0 exponentially, whence the result.

For the corollary, observe that

$$\dot{y}(t) \cdot 3y^2(t) \int_{-\infty}^t a(t-s) \frac{1}{y^2(s)} = - \int_{-\infty}^t a'(t-s) \left[ \frac{y^3(t)}{y^2(s)} - y(s) \right] ds .$$

Using this, one can apply an argument analogous to the previous one.

Finally, we want to prove that solutions depend continuously on  $\mu$ , even at  $\mu = 0$ . Let  $f \in Y^{\sigma, n}$  be given such that it either has a small norm or it satisfies the conditions of theorem 2.4. We know that a unique solution  $y(t)$  satisfying  $y(-\infty) = 1$  exists both for  $\mu = 0$  and for  $\mu > 0$ . In (2.1), we put  $g = \sum_{i=1}^N g_i$ ,  $h = \sum_{i=1}^N h_i$ , and  $z = y - \sqrt[3]{\frac{h}{g}}$  (for  $\mu = 0$ ,  $f = 0$ , the first equation of (2.1) is solved by  $y = \sqrt[3]{\frac{h}{g}}$ ). We obtain

$$\begin{aligned} -\mu \dot{z} &= g \left( \sqrt[3]{\frac{h}{g}} + z \right)^3 - h - f(t) \left( \sqrt[3]{\frac{h}{g}} + z \right)^\alpha + \mu \frac{d}{dt} \sqrt[3]{\frac{h}{g}} \\ (2.5) \quad \dot{g}_i &= -\lambda_i g_i + \frac{\kappa_i}{\left( \sqrt[3]{\frac{h}{g}} + z \right)^2} \\ \dot{h}_i &= -\lambda_i h_i + \kappa_i \left( \sqrt[3]{\frac{h}{g}} + z \right) . \end{aligned}$$

As we have proved, there exists some  $\mu_0 > 0$  such that for every  $\mu \in [0, \mu_0]$

system (2.5) has a unique solution in the Banach manifold

$$M_n = \{(z, g_i, h_i) | z \in Y^{\sigma, n}, g_i = \frac{\kappa_i}{\lambda_i \sqrt[3]{\frac{h}{g}}^2} \in Y^{\sigma, n}, h_i = \frac{\kappa_i}{\lambda_i} \sqrt[3]{\frac{h}{g}} \in Y^{\sigma, n}\}. \quad \text{In}$$

particular, let  $z_0, g_{i,0}, h_{i,0}$  denote the solution for  $\mu = 0$ .

Linearizing at this solution (or likewise at any solution for  $\mu > 0$ ), we obtain a system of linear ODE's with a matrix approaching a constant limit as  $t \rightarrow -\infty$  and  $t \rightarrow +\infty$ . From a discussion of the asymptotic behavior of solutions of the linearized system for  $t \rightarrow \pm\infty$ , one can easily see that for

any inhomogeneity in  $(Y^{\sigma,n})^{2N+1}$  there is a unique solution in the tangent space of  $M_n$ . The argument parallels our existence proof for solutions: First consider the problem on  $(-\infty, -T]$  with  $T$  large, where the matrix is approximated by the linearization at the trivial solution. Continuation of solutions for  $t > -T$  presents no problem, since the equation is linear, and finally the behavior for  $t \rightarrow +\infty$  must be discussed. We leave the details of the analysis to the reader.

Thus the linearization is a densely defined bijective operator from the tangent space of  $M_n$  onto  $(Y^{\sigma,n})^{2N+1}$ . It is thus natural to attempt proving the existence of a continuous family of solutions in a neighborhood of  $\mu = 0$  using the implicit function theorem. One does, however, face the problem that the term  $\dot{\mu}z$  represents an unbounded operator.

The first equation of (2.4) has the form

$$-\mu \frac{d}{dt} (z-z_0) = \rho(t)(z-z_0) - f(t) \cdot L(h-h_0, f-f_0) + \text{nonlinear terms} + o(\mu)$$

where  $\rho(t) = 3g_0 \left( \sqrt[3]{\frac{h_0}{g_0}} + z_0 \right)^2 - \alpha f(t) \left( \sqrt[3]{\frac{h_0}{g_0}} + z_0 \right)^{\alpha-1}$  is positive and

converges to  $\sum_{i=1}^N \frac{K_i}{\lambda_i}$  for  $t \rightarrow \infty$ .  $L$  is linear in its arguments, and the term  $o(\mu)$  does not involve any unbounded operators, after the second and third

equation of (2.4) have been substituted into the first to replace  $\dot{g}$  and  $\dot{h}$ .

It is easy to show that the operator  $(\mu \frac{d}{dt} + \rho(t))^{-1} : Y^{\sigma,n} \rightarrow Y^{\sigma,n}$  is strongly continuous with respect to  $\mu$ . Denoting  $V = (z-z_0, g_1-g_1, 0, \dots, g_N-g_N, h_1-h_1, 0, \dots, h_N-h_N, 0)$ , we can thus rewrite (2.5) in the abstract form.

$$(2.6) \quad L(\mu)V = N(\mu, V) \iff V - (L(\mu))^{-1}N(\mu, V) = 0$$

where  $L(\mu)$  has a strongly continuous inverse and  $N(0, 0) = 0$ ,  $D_V N(0, 0) = 0$ .

The existence of a continuous solution  $V(\mu)$  now follows from the following theorem.

Theorem 2.8:

Let  $X, Y$  and  $Z$  be Banach spaces,  $U$  a neighborhood of  $(0,0)$  in  $X \times Y$ , and  $F: U \rightarrow Z$  a mapping having the following properties:

- (i)  $F(0,0) = 0$
- (ii)  $F$  is continuous
- (iii)  $F$  is continuously differentiable with respect to  $y$  for each fixed  $x$ .
- (iv)  $D_y F(0,0): Y \rightarrow Z$  is an isomorphism.
- (v)  $D_y F$  is continuous at the point  $(0,0)$ .

Then the equation  $F(x,y) = 0$  has a unique resolution  $y = f(x)$  in some neighborhood of  $(0,0)$ , and  $f$  is continuous.

The proof of this theorem differs by no means from the standard proof of the implicit function theorem (cf. [10], [11]), but it is crucial for our problem that (iii) and (v) are sufficient rather than continuity of  $D_y F$  in a neighborhood of  $(0,0)$  as usually required. Namely, we can identify  $X$  with  $\mathbb{R}$ ,  $Y$  with the tangent space of  $M_n$ ,  $Z$  with  $Y^{\sigma, n}$ ,  $x$  with  $\mu$  and  $y$  with  $V$ . For  $\mu$  fixed, the term  $L(\mu)^{-1}N(\mu, V)$  depends smoothly on  $V$ , moreover, since  $\lim_{\mu \rightarrow 0, V \rightarrow 0} D_V N(\mu, V) = 0$ , we also have

$$\lim_{\mu \rightarrow 0, V \rightarrow 0} D_V (L(\mu)^{-1}N(\mu, V)) = \lim_{\mu \rightarrow 0, V \rightarrow 0} L(\mu)^{-1} D_V N(\mu, V) = 0. \text{ Hence Theorem 2.8}$$

applies to (2.6), although the standard form of the implicit function theorem would not. This yields a continuous solution  $V = V(\mu)$ .

Moreover, the mapping  $(\mu, z) \mapsto (\mu \frac{d}{dt} + \rho(t))^{-1}z$  is a  $C^k$ -mapping from  $\mathbb{R} \times Y^{\sigma, n}$  into  $Y^{\sigma, n-k}$ . From the following theorem, which was also proved in [10], [11], one concludes that  $V(\mu)$  is actually a  $C^k$ -function of  $\mu$  when regarded as lying in  $Y^{\sigma, n-k}$ .

Theorem 2.9:

Let  $Y^{(k)}$  and  $Z^{(k)}$  resp. ( $k=0, 1, \dots, N$ ) be two hierarchies of Banach spaces such that  $Y^{(k)} \subset Y^{(k+1)}$ ,  $Z^{(k)} \subset Z^{(k+1)}$ , the imbeddings being continuous. Let  $X$  be a finite dimensional Banach space and  $F$  a mapping from a neighborhood  $U$  of 0 in  $X \times Y^{(N)}$  into  $Z^{(N)}$  having the following properties:

- (i)  $F(U \cap (X \times Y^{(k)})) \subset Z^{(k)}$   $k=0, 1, \dots, N$
- (ii) For each fixed  $k$ ,  $F_k := F|_{U \cap (X \times Y^{(k)})}$  satisfies the conditions of Theorem 2.8, when it is considered as a mapping from  $X \times Y^{(k)}$  into  $Z^{(k)}$ . For  $x$  fixed,  $F_k(x, \cdot)$  is a smooth (i.e. sufficiently often differentiable) mapping.
- (iii)  $F: X \times Y^{(k)} \rightarrow Z^{(k+m)}$  is of class  $C^m$  for each  $k=0, 1, \dots, N$  and  $m < N-k$ .
- (iv) The mapping  $(x, y, u^1, \dots, u^j) \mapsto z = D_{x,y} F(x, y)(u^1, \dots, u^j)$  is continuous from  $X \times Y^{(k)} \times (Y^{(k)})^j$  into  $L^1(X, Y^{(k+j)})$ .

Then the solution  $y = f(x) \in Y^{(0)}$  existing by theorem 2.8 is a  $C^m$ -function of  $x$  in some neighborhood  $V_m$  of 0, if  $y$  is regarded as an element of  $Y^{(m)}$ .

We summarize our results in the following:

Theorem 2.10:

Let  $f \in Y^{\sigma, n}$  be given such that either  $f$  has small norm or  $f(t) \equiv 0$  for  $t$  greater than some  $t_0 < \infty$ . Then, for each  $\mu \in [0, \infty]$ , (1.1) has a unique solution  $y$  satisfying  $y - 1 \in X^{\sigma, n}$ . In the limit  $\mu \rightarrow 0$ ,  $y - 1 \in X^{\sigma, n}$  depends continuously on  $\mu$ , and it is a  $C^k$ -function of  $\mu$  when regarded as dwelling in  $X^{\sigma, n-k}$ .

### 3. The Discretization Scheme

When solving (1.1) numerically, one faces the problem that it is to be solved on an infinite interval. A reasonable way of doing this is to cut at  $-T \ll 0$ , and replace  $y(t)$  for  $t < -T$  by its limit  $\lim_{t \rightarrow -\infty} y(t) = 1$ . We thus obtain the approximating problem

$$(3.1) \quad \begin{aligned} \mu \dot{y}_{-T} + \int_{-\infty}^{-T} a(t-s) ds \cdot (y_{-T}^3(t) - 1) + \int_{-T}^t a(t-s) \left( \frac{y_{-T}^3(t)}{y_{-T}^2(s)} - y_{-T}(s) \right) ds - \\ - f(t) y_{-T}^\alpha(t) = 0 \quad , \end{aligned}$$

$$(3.2) \quad y_{-T}(t) - 1 = 0, \quad t < -T \quad .$$

On the finite interval the integrodifferential equation can now be discretized in a straightforward manner. Like Nevanlinna [8], we use a first order implicit (Euler-type) method, because for this simple procedure we can prove that the qualitative properties of solutions of (1.1), such as exponential decay at infinity and uniform convergence as  $\mu \rightarrow 0$ , carry over to the discrete problem. Since these properties are essential for the continuous problems it is very reasonable to require that the computed approximating solutions exhibit them too. Our computations have shown that good approximations can be obtained with quite large mesh sizes, and so the computational effort for the first order scheme remains reasonably small.

We choose mesh points  $t_i = ih$ ,  $i \in \mathbb{Z}$ , where  $t_{-m} = -T$ , and denote by  $y_i$  the approximation to  $y(t_i)$  (or, respectively, to  $y_{-T}(t_i)$ ). Then our discretized form of the equation reads

$$(3.3) \quad \begin{aligned} \mu \frac{y_i - y_{i-1}}{h} + \int_{-\infty}^{t_{-m}} a(t_i - s) ds \cdot (y_i^3 - 1) + h \sum_{j=-m+1}^i a((i-j)h) \left( \frac{y_i^3}{y_j^2} - y_j \right) - \\ - f(t_i) y_i^\alpha = 0 \quad , \quad i > -m \end{aligned}$$

$$(3.4) \quad y_i - 1 = 0, \quad i < -m.$$

Obviously,

$$(3.5) \quad \int_{-\infty}^{t_{-m}} a(t_i - s) ds = \sum_{\ell=1}^N \frac{\kappa_\ell}{\lambda_\ell} e^{\lambda_\ell(t_{-m} - t_i)}.$$

Equation (3.3) has the form

$$(3.6) \quad c_1 y_i^3 + c_2 y_i^\alpha + c_3 y_i = c_4,$$

where the  $c$ 's depend on  $\mu$ ,  $h$ ,  $t_{-m}$ ,  $t_i$  and  $y_j$ ,  $j < i$ .\*

The analysis of the discrete equation will be carried out in the same sort of spaces as the analysis of the continuous equation. We therefore define discrete analogues of the exponentially weighted spaces introduced in Definition 2.2.

Definition 3.1:

Let  $x_h^\sigma := \{\hat{f} = (f_i)_{i=-\infty}^{\infty} e^{\lambda^\infty} \lim_{i \rightarrow \infty} f_i =: f_\infty \text{ exists,}$

$$\lim_{i \rightarrow \infty} e^{i\sigma h} |f_i - f_\infty| = 0, \lim_{i \rightarrow -\infty} e^{-i\sigma h} |f_i| = 0\}$$

and  $y_h^\sigma := \{\hat{g} = (g_i)_{i=-\infty}^{\infty} e^{\lambda^\infty} \lim_{i \rightarrow \infty} e^{i\sigma h} |g_i| = \lim_{i \rightarrow -\infty} e^{-i\sigma h} |g_i| = 0\}$ . The

natural norms in these spaces are

$$\|\hat{f}\|_{x_h^\sigma} = \sup_{i>0} e^{i\sigma h} |f_i - f_\infty| + \sup_{i<0} e^{-i\sigma h} |f_i| + |f_\infty|$$

and

$$\|\hat{g}\|_{y_h^\sigma} = \sup_{i>0} e^{i\sigma h} |g_i| + \sup_{i<0} e^{-i\sigma h} |g_i|.$$

Setting  $\hat{z} = (y_i - 1)_{i=-\infty}^{\infty}$ , we rewrite (3.3) and (3.4) in the abstract form

\* Equation (3.6) will be discussed at the end of this chapter.

$$(3.7) \quad F_{h,m}(\hat{z}) = 0 .$$

Now let  $f \in Y^{\sigma,n}$  ( $\sigma > 0$ ,  $n \in \mathbb{N}$ ) be given such that the assumptions of theorem

2.4 hold. It is an easy exercise to show that, for any  $\varepsilon \in [0, \sigma]$ ,

$$(3.8) \quad F_{h,m} : X_h^{\sigma-\varepsilon} \rightarrow Y_h^{\sigma-\varepsilon}$$

(we explain below why  $\varepsilon$  is introduced).

The aim of the following analysis is to prove that  $(y_i)_{i=-\infty}^{\infty}$  converges to  $(y(t_i))_{i=-\infty}^{\infty}$  in the topology of  $X_h^{\sigma-\varepsilon}$ . The proof will be based on Keller's [3] nonlinear stability-consistency concept.

Let us first show consistency. The local discretization error

$\ell = (\ell_i)_{i=-\infty}^{\infty}$  is defined by

$$(3.9) \quad \ell = F_{h,m}((y(t_i) - 1)_{i=-\infty}^{\infty}) .$$

For  $f \in Y^{\sigma,1}$  (which implies  $\mu y \in Y^{\sigma,2}$ , uniformly in  $\mu$ ), we find, using the exponential decay of  $y$  as  $t \rightarrow \pm\infty$ , that in the limit  $t_{-m} \rightarrow -\infty$ ,  $h \rightarrow 0$

$$(3.10) \quad (a) \quad |\ell_i| \leq o(1)e^{-\sigma|t_i|}, \quad i < -m ,$$

$$(b) \quad |\mu \frac{y(t_i) - y(t_{i-1})}{h} - \mu y'(t_i)| = \text{const. } o(1)h e^{-\sigma|t_i|} ,$$

$$(c) \quad \left| \int_{-\infty}^{t_{-m}} a(t_i - s) ds \cdot (y^3(t_i) - 1) - \int_{-\infty}^{t_{-m}} a(t_i - s) \left( \frac{y^3(t_i)}{y^2(s)} - y(s) \right) ds \right|$$

$$\leq \text{const. } o(1) e^{\sigma(2t_{-m} - t_i)} ,$$

$$(d) \quad \left| h \sum_{j=-m+1}^i a((i-j)h) \left( \frac{y^3(t_i)}{y^2(t_j)} - y(t_j) \right) - \int_{t_{-m}}^{t_i} a(t_i - s) \left( \frac{y^3(t_i)}{y^2(s)} - y(s) \right) ds \right|$$

$$\leq \text{const. } o(1) h e^{-\sigma|t_i|} .$$

Here  $o(1)$  stands for a factor that vanishes as  $t_i \rightarrow -\infty$ . Therefore,

$$(3.11) \quad |\ell_i| < \text{const. } o(1) \begin{cases} h e^{-\sigma|t_i|} + e^{-\sigma|t_i-2t_{-m}|}, & i > -m \\ e^{-\sigma|t_i|}, & i \leq -m \end{cases}.$$

From definition 3.1 we conclude that

$$(3.12) \quad \| \ell \|_{Y_h^{\sigma-\varepsilon}} < \text{const. } (h + o(e^{-\varepsilon|t_{-m}|})) .$$

The constant is independent of  $h$ ,  $t_{-m}$ ,  $0 < \mu < \infty$ ,  $0 < \varepsilon < \varepsilon_0 < \sigma$ . Note that,

in particular, the error estimate contains a term  $o(e^{-\varepsilon|t_{-m}|})$ . The reason for this is that, when approximating (1.1) by (3.1), (3.2), we have replaced  $f$  by 0 for  $t < -T$ , and in the norm of  $Y^{0-\varepsilon, n}$  this introduces an error of the order  $o(e^{-\varepsilon|T|})$ . This is the reason why we have introduced the  $\varepsilon$ ; for  $\varepsilon = 0$  we would still get convergence, but no estimate for the order.

(3.12) settles consistency.

For the stability analysis, we calculate the Fréchet derivative of  $F_{h,m}$  at the exact solution  $(y(t_i) - 1)_{i=-\infty}^{\infty}$ , which is denoted by

$$(3.13) \quad L_{h,m} := D_{\hat{z}} F_{h,m} ((y(t_i) - 1)_{i=-\infty}^{\infty}) .$$

For  $\hat{u} = (u_i)_{i=-\infty}^{\infty} \in X_h^{\sigma-\varepsilon}$  we obtain

$$(3.14) \quad (L_{h,m}\hat{u})_i = \begin{cases} u_i, & i < -m \\ \mu \frac{u_i - u_{i-1}}{h} + 3 \int_{-\infty}^{t_{-m}} a(t_i - s) ds y(t_i)^2 u_i \\ + 3h \left( \sum_{j=-m+1}^i a((i-j)h) \frac{y^2(t_i)}{y^2(t_j)} \right) u_i \\ - h \sum_{j=-m+1}^i a((i-j)h) \left( 2 \frac{y^3(t_i)}{y^3(t_j)} + 1 \right) u_j \\ - \alpha f(t_i) y(t_i)^{\alpha-1} u_i, & i > -m \end{cases} .$$

Stability means that  $L_{h,m}^{-1}$  exists and that it is bounded as an operator from  $y_h^{\sigma-\varepsilon}$  into  $x_h^{\sigma-\varepsilon}$  uniformly with respect to  $h, t_{-m}, \mu$  and  $0 < \varepsilon < \varepsilon_0 < \sigma$ .

Therefore we look at the equation  $L_{h,m}\hat{u} = \hat{v}$  for  $\hat{v} = (v_i)_{i=-\infty}^{\infty} \in y_h^{\sigma-\varepsilon}$ .

For  $i < -m$  we find  $u_i = v_i$ , and for  $i > -m$  we show that

$$(3.15) \quad G_i(h, -m, \mu) = \frac{\mu}{h} + 3 \int_{-\infty}^{t_{-m}} a(t_i - s) y^2(t_i) ds + 3h \sum_{j=-m+1}^{i-1} a((i-j)h) \frac{y^2(t_i)}{y^2(t_j)} - \alpha f(t_i) v^{\alpha-1}(t_i) ,$$

which is the coefficient of  $u_i$  in (3.14), is bounded away from 0 uniformly in  $h, t_{-m}$  and  $\mu$ .

It is easy to show that

$$(3.16) \quad G_i(h, -m, \mu) = \frac{\mu}{h} + 3 \int_{-\infty}^{t_i} a(t_i - s) \frac{y^2(t_i)}{y^2(s)} ds - \alpha f(t_i) y^{\alpha-1}(t_i) + O(h) + O(e^{-\sigma(|t_i| + |t_{-m}|)}) .$$

Using (1.1) we get

$$\begin{aligned}
 G_i(h, -m, \mu) &= \frac{\mu}{h} + \frac{3}{y(t_i)} \left( \int_{-\infty}^{t_i} a(t_i - s) y(s) ds - \dot{u}y(t_i) + \right. \\
 (3.17) \quad &\quad \left. + \left( 1 - \frac{\alpha}{3} \right) f(t) y^\alpha(t_i) \right) + O(h) + O(e^{-\sigma(|t_i| + |t_{-m}|)}) .
 \end{aligned}$$

It follows from chapter 2 that

$$(3.18) \quad 0 < \underline{y}_0 < y(t, \mu) < \overline{y}_0, \quad |\dot{y}(t, \mu)| < \tilde{y}$$

uniformly for  $\mu \in [0, \infty)$ . For  $f < 0$ , (3.16) provides a uniform lower bound for  $G_i$ , and, for  $f > 0$ , (3.18) provides a uniform lower bound, since  $\alpha < 3$ .

The preceding considerations make it apparent, why the term  $\int_{-\infty}^{t_{-m}} a(t_i - s) ds (y_i^3 - 1)$  should be maintained in (3.3). If this term were neglected, the uniform bounds on  $G_i$  would no longer hold, and, unless a constraint of the form  $\frac{\mu}{h} > \text{const.}$  is imposed, an artificial boundary layer can be generated at  $t_{-m}$ . We see from the above that  $L_{h,m}$  can formally be inverted. It remains to be proved that the solution  $\hat{u}$  of

$$(3.19) \quad L_{h,m} \hat{u} = \hat{v}$$

satisfies an estimate

$$(3.20) \quad \|\hat{u}\|_{x_h^{\sigma-\varepsilon}} \leq \text{const.} \quad \|\hat{v}\|_{y_h^{\sigma-\varepsilon}}$$

with the constant independent of  $h, t_{-m}, \mu$  and  $\varepsilon$ .

We begin with the reduced problem for  $\mu = 0$ . (3.14) yields

$$(3.21) \quad u_i = h \sum_{j=-m+1}^{i-1} a((i-j)h) a_{i,j}^{(h, -m)} u_j + v_i^0$$

where

$$(3.22) \quad a_{i,j}(h, -m) = \frac{2 \frac{y^3(t_i)}{y^3(t_j)} + 1}{G_i(h, -m, 0)}, \quad v_i^0 = \frac{v_i}{G_i(h, -m, 0)}$$

with  $G_i$  as defined in (3.17). Since  $y(-\infty) = 1$ , this implies, after a simple calculation

$$a_{i,j}(h, -m) = \frac{1}{h \sum_{\ell=1}^N \frac{K_\ell}{e^{\lambda_\ell h} - 1}} + \gamma = : C(h, \gamma)$$

where  $\gamma \rightarrow 0$  as  $h \rightarrow 0$ ,  $t_{-m} \rightarrow -\infty$ ,  $t_i \rightarrow -\infty$ .

Using the form of  $a$ , we get from (3.21)

$$(3.24) \quad |u_i| \leq C(h, \gamma) h \sum_{j=-m+1}^{i-1} \sum_{\ell=1}^N K_\ell e^{-\lambda_\ell (t_i - t_j)} |u_j| + |v_i^0| .$$

The solution  $w_i$  of the equation obtained by replacing  $|v_i|$  by the larger quantity  $e^{i(\sigma-\varepsilon)h} \|\bar{v}\|_\ell^\infty$ , where

$$(3.25) \quad v_i^0 = e^{i(\sigma-\varepsilon)h} \bar{v}_i, \quad \bar{v} = (\bar{v}_i)_{i=-\infty}^\infty e^{-\lambda^\infty}$$

provides an upper bound for  $|u_i|$ . In analogy to chapter 2, we substitute

$$(3.26) \quad g_i^\ell = K_\ell h \sum_{j=-m+1}^{i-1} e^{-\lambda_\ell (t_i - t_j)} w_j, \quad \ell = 1, 2, \dots, N ,$$

which yields the difference equation

$$(3.27) \quad g_{i+1}^{\ell} - g_i^{\ell} = K_{\ell} h e^{-\lambda_{\ell} h} w_i + (e^{-\lambda_{\ell} h} - 1) g_i^{\ell}$$

and the following difference equation for  $w_i$

$$(3.28) \quad w_{i+1} - w_i = C(h, \gamma) \sum_{\ell=1}^N (e^{-\lambda_{\ell} h} - 1) g_i^{\ell} + C(h, \gamma) \sum_{\ell=1}^N K_{\ell} h e^{-\lambda_{\ell} h} w_i + (e^{(\sigma-\varepsilon)h} - 1) e^{i(\sigma-\varepsilon)h} \|\bar{v}\|_{\ell^{\infty}}$$

(3.27) and (3.28) form a system of difference equations. Setting

$$(3.29) \quad z_i = (w_i, g_i^1, \dots, g_i^N), \quad \varphi_i = ((e^{(\sigma-\varepsilon)h} - 1) e^{i(\sigma-\varepsilon)h} \|\bar{v}\|_{\ell^{\infty}}, 0, \dots, 0),$$

we can rewrite this system in the form

$$(3.30) \quad z_{i+1} = (I + A(h, \gamma)) z_i + \varphi_i \quad i \geq -m$$

where

$$(3.31) \quad A(h, \gamma) = \begin{pmatrix} C(h, \gamma) \sum_{\ell=1}^N K_{\ell} h e^{-\lambda_{\ell} h} & C(h, \gamma) (e^{-\lambda_1 h} - 1) \dots C(h, \gamma) (e^{-\lambda_N h} - 1) \\ K_1 h e^{-\lambda_1 h} & e^{-\lambda_1 h} - 1 \\ \vdots & \ddots \\ K_N h e^{-\lambda_N h} & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ & & & & & & & e^{-\lambda_N h} - 1 \end{pmatrix}$$

The solution of (3.30) is given by

$$(3.32) \quad z_i = (I + A(h, \gamma))^{i-1+m} z_{-m+1} + \sum_{j=-m}^{i-1} (I + A(h, \gamma))^{i-j-1} \varphi_j$$

with the initial condition  $z_{-m+1} = (e^{-(m-1)(\sigma-\varepsilon)h} \bar{v})_{\ell^\infty}, 0, \dots, 0$ .

The goal of the following analysis is to show that

$$(3.33) \quad \sup_{i \leq 0} e^{-i(\sigma-\varepsilon)h} \|z_i\| \leq \text{const.} \|\bar{v}\|_{\ell^\infty}$$

which implies

$$(3.34) \quad \sup_{i \leq 0} e^{-i(\sigma-\varepsilon)h} |u_i| \leq \text{const.} \sup_{i \leq 0} e^{-i(\sigma-\varepsilon)h} |\bar{v}_i| .$$

Summing up the geometric series in (3.32), we obtain the estimate

$$(3.35) \quad e^{-i(\sigma-\varepsilon)h} \|z_i\| \leq \text{const.} \left[ \frac{\|(I + A(h, \gamma))^{i+m-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\|}{e^{(i+m-1)(\sigma-\varepsilon)h}} + 1 \right] \|\bar{v}\|_{\ell^\infty} \cdot \left[ \left( (e^{(\sigma-\varepsilon)h} - 1) \left\| \left( I - \frac{I + A(h, \gamma)}{e^{(\sigma-\varepsilon)h}} \right)^{-1} \right\| \right) \right].$$

We thus have to prove estimates of the following form

$$(3.36) \quad \left\| \left( \frac{I + A(h, \gamma)}{e^{(\sigma-\varepsilon)h}} \right)^k \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\| \leq \text{const.}, \quad k \in \mathbb{N}$$

and

$$(3.37) \quad |e^{(\sigma-\epsilon)h} - 1| + \left\| \left( I - \frac{I+A(h, \gamma)}{e^{(\sigma-\epsilon)h}} \right)^{-1} \right\| < \text{const.}$$

Both will follow from an analysis of the Jordan form of  $A(h, \gamma)$ . For  $h, \gamma$  small enough we can write

$$(3.38) \quad A(h, \gamma) = h \left( \frac{d}{dh} A(0, 0) + O(h) + O(\gamma) \right)$$

where

$$(3.39) \quad \frac{d}{dh} A(0, 0) = \begin{pmatrix} C(0, 0) & \sum_{k=1}^N K_k & -C(0, 0)\lambda_1 & \dots & -C(0, 0)\lambda_N \\ K_1 & -\lambda_1 & & & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & \ddots & \ddots & \ddots & \lambda_N \\ K_N & & & & \end{pmatrix}.$$

This matrix has the characteristic polynomial

$$(3.40) \quad p(\rho) = C(0, 0) \sum_{k=1}^N K_k - \rho - C(0, 0) \sum_{k=1}^N \frac{\lambda_k K_k}{\lambda_k + \rho}.$$

Recalling that  $C(0, 0) = \left( \sum_{k=1}^N \frac{K_k}{\lambda_k} \right)^{-1}$ , it is an easy exercise to show that the

root 0 is two fold. An analysis similar to that given for (2.3) shows that all remaining zeros are real and negative.

A similar calculation shows that zero is also a double eigenvalue of  $\frac{d}{dh} A(0, 0) + O(h)$  (as of (3.38)). Therefore we get for the eigenvalues of  $A(h, \gamma)$

$$(3.41)(a) \quad \rho_1(h, \gamma) = h o(1), \quad \rho_2(h, \gamma) = h o(1) \quad \text{as } \gamma \rightarrow 0$$

$$(3.41)(b) \quad \rho_i(h, \gamma) = h(\tilde{\rho}_i + o(1)) \quad \text{as } h, \gamma \rightarrow 0, \quad i = 3(1)(N+1)$$

where  $\tilde{\rho}_i < 0$  for  $i = 3(1)(N+1)$  such that

$$(3.42) \quad \left| \frac{1+\rho_i(h, \gamma)}{e^{(\sigma-\varepsilon)h}} \right| < 1,$$

holds. Equality in (3.42) only holds for  $h = 0$ . Since  $\frac{1}{h} A(h, 0)$  is holomorphic in  $h = 0$  and since the eigenvalues of  $\frac{1}{h} A(h, 0)$  do not change multiplicities as  $h \rightarrow 0$  (the negative eigenvalues of  $\frac{d}{dh} A(0, 0)$  are distinct and 0 is a double eigenvalue of  $\frac{1}{h} A(h, 0)$ ). There is a matrix  $G(h)$  such that  $G(h), G^{-1}(h)$  are holomorphic in  $h = 0$  and  $J(h)$  defined by

$$A(h, 0) = G(h)J(h)G^{-1}(h)$$

is the Jordan form of  $A(h, 0)$  (see [2]). Therefore (3.36), (3.37) hold for  $\gamma = 0$ . A simple perturbation argument assures (3.36), (3.37) for  $\gamma$  sufficiently small. Thus for  $h$  sufficiently small and  $K > 0$  sufficiently large, we have proved that

$$(3.43) \quad \sup_{\substack{t_i < -K \\ t_i}} e^{-t_i(\sigma-\varepsilon)} |u_i| \leq \text{const} \|\bar{v}\|_\infty.$$

The solution  $u_i$  can be continued over the finite interval  $[-K, 0]$  and by a standard stability analysis (see [1]) we obtain (3.34).

We now have to treat the case  $t_i > 0$ . For this, we rewrite (3.21) as

$$(3.44) \quad \begin{aligned} u_i &= h \sum_{j=-m+1}^{i-1} a((i-j)h) \alpha_{i,j}(h, -m) u_j \\ &+ h \sum_{j=I}^{i-1} a((i-j)h) \alpha_{i,j}(h, -m) u_j + v_i \end{aligned}$$

where it is assumed that  $t_I > K$  is sufficiently large. After some

calculation, we get from (3.15), (3.22).

$$(3.45) \quad \alpha_{i,j}(h, -m) = D(h) + \beta_{ij}(h, -m)$$

where

$$(3.46) \quad D(h) = \frac{1}{\sum_{\ell=1}^N \frac{K_{\ell}}{\frac{\lambda_{\ell} h}{e^{\lambda_{\ell} h} - 1}}} ; \quad |\beta_{i,j}(h, -m)| = O(e^{-\sigma t_j}), \quad t_j > K.$$

It is therefore natural to study the equation

$$(3.47) \quad \tilde{u}_i = D(h) \sum_{j=1}^{i-1} a((i-j)h) \tilde{u}_j + \tilde{v}_i ,$$

where  $\tilde{v}_i$  is  $v_i$  plus the first sum in (3.44), and interpret (3.44) as a perturbation of (3.47). As before we put

$$(3.48) \quad q_i^{\ell} = K_{\ell} h \sum_{j=1}^{i-1} e^{-\lambda_{\ell}(t_i - t_j)} \tilde{u}_j$$

which leads to the difference equation

$$(3.49) \quad q_{i+1}^{\ell} - q_i^{\ell} = K_{\ell} h e^{-\lambda_{\ell} h} \sum_{j=1}^N q_i^j + (e^{-\lambda_{\ell} h} - 1) q_i^{\ell} \\ + h K_{\ell} e^{-\lambda_{\ell} h} \tilde{v}_i .$$

Here the relation

$$(3.50) \quad \tilde{u}_i = D(h) \sum_{j=1}^N q_i^j + \tilde{v}_i$$

has been used.

Putting  $g_i = (g_i^1, \dots, g_i^N)$ ,  $e(h) := (K_1 e^{-\lambda_1 h}, \dots, K_N e^{-\lambda_N h})$ , we obtain the following matrix form of (3.49)

$$(3.51) \quad g_{i+1} = (I + B(h))g_i + \tilde{v}_i h \cdot e(h) .$$

This has the solution

$$(3.52) \quad g_i = h \sum_{j=1}^{i-1} (I + B(h))^{i-j-1} \tilde{v}_j e(h) .$$

When dealing with the case  $t_i < 0$ , we used a 'redundant' system of equations rather than an analogue of (3.49). The reason for this was that it is easier to compute the characteristic polynomial of the matrix of the 'redundant' system. In the 'redundant' ((N+1)-dimensional instead of N-dimensional) form (3.49) reads

$$(3.53) \quad \begin{aligned} \tilde{u}_{i+1} - \tilde{u}_i &= D(h)h \sum_{\ell=1}^N K_\ell e^{-\lambda_\ell h} \tilde{u}_i + D(h) \sum_{\ell=1}^N (e^{-\lambda_\ell h} - 1) g_i^\ell \\ &\quad + \tilde{v}_{i+1} - \tilde{v}_i \\ g_{i+1}^\ell - g_i^\ell &= K_\ell h e^{-\lambda_\ell h} \tilde{u}_i + (e^{-\lambda_\ell h} - 1) g_i^\ell . \end{aligned}$$

When we write this in matrix form

$$(3.54) \quad \tilde{z}_{i+1} = (I + A(h))\tilde{z}_i + d_i$$

(where  $z_i = (\tilde{u}_i, g_i^1, \dots, g_i^N)$ ,  $d_i = (\tilde{v}_{i+1} - \tilde{v}_i, 0, \dots, 0)$ ), we see immediately that  $A(h)$  is the same matrix as (3.31), except that  $C(h, \gamma)$  is replaced by  $D(h)$ . The characteristic polynomial is

$$(3.55) \quad q(\rho) = D(h)h \sum_{\ell=1}^N K_\ell e^{-\lambda_\ell h} - \rho + D(h)h \sum_{\ell=1}^N K_\ell \frac{e^{-\lambda_\ell h} (e^{-\lambda_\ell h} - 1)}{\rho - (e^{-\lambda_\ell h} - 1)} .$$

It is easily verified that  $\rho = 0$  is a double root. Moreover, since

$$(3.56) \quad D(h) = C(0,0) + O(h) ,$$

the other roots are small perturbations of those of  $p(\rho)$  as given by (3.40) and have therefore negative real parts. When passing from  $A(h)$  to the  $N$ -dimensional matrix  $B(h)$ , the eigenvalues obviously remain the same, except that 0 as an eigenvalue of  $B(h)$  has multiplicity one rather than two.

Hence there is a matrix  $E(h)$  such that  $E(h), E^{-1}(h)$  are continuous for  $h \in [0, h_0]$  and the Jordan form  $J(h)$  of  $B(h)$

$$(3.57) \quad J(h) = E^{-1}(h)B(h)E(h)$$

has the block form

$$(3.58) \quad J(h) = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ 0 & & & \\ 0 & & J_{-}(h) & \\ \vdots & & & \\ 0 & & & \end{pmatrix} ,$$

where the  $(N-1) \times (N-1)$ -matrix  $J_{-}(h)$  has only eigenvalues with negative real parts. The continuity of  $E(h), E^{-1}(h)$  holds since  $\frac{1}{h} B(h)$  is analytic in  $h$  and since the eigenvalues of  $\frac{1}{h} B(h)$  are distinct even for  $h = 0$  (see [2]).

If we put  $q_i = E(h)w_i$ , (3.52) yields

$$(3.59) \quad w_i = h \sum_{j=I}^{i-1} (I + J(h))^{i-j-1} \tilde{v}_j E^{-1}(h) e(h) .$$

In the first component this reads in particular

$$(3.60) \quad w_i^1 = h \sum_{j=I}^{i-1} \tilde{v}_j (E^{-1}(h) e(h))^1 .$$

From this we obtain the following estimate for  $t_I \rightarrow \infty$ :

$$\begin{aligned}
 (3.61) \quad & \lim_{i \rightarrow \infty} |w_i^1| \leq \text{const} \sup_{i > I} e^{i(\sigma-\varepsilon)h} |\tilde{v}_i| \\
 & \sup_{i > I} e^{i(\sigma-\varepsilon)h} |w_i^1 - \lim_{i \rightarrow \infty} w_i^1| \leq \text{const. } o(1) \sup_{i > I} e^{i(\sigma-\varepsilon)h} |\tilde{v}_i| .
 \end{aligned}$$

For the components  $(w_i^2, \dots, w_i^N)$ , where only eigenvalues with negative real parts are involved, an analogous estimate follows from the same arguments that have been used in the case  $t_i < 0$ , and we even have  $\lim_{i \rightarrow \infty} w_i^\ell = 0$  for  $\ell > 1$ .

Let us now introduce the spaces

$$A_{h,I}^{\sigma-\varepsilon} = \{f = (f_i)_{i=I}^\infty \mid \lim_{i > I} e^{i(\sigma-\varepsilon)h} |f_i| = 0\}$$

$$B_{h,I}^{\sigma-\varepsilon} = \{f = (f_i)_{i=I}^\infty \mid \lim_{i \rightarrow \infty} f_i = : f_\infty \text{ exists, } \lim_{i > I} e^{i(\sigma-\varepsilon)h} |f_i - f_\infty| = 0\}$$

and the operator

$$(3.62) \quad P_I(h) : A_{h,I}^{\sigma-\varepsilon} \rightarrow B_{h,I}^{\sigma-\varepsilon}$$

which is defined as the solution operator corresponding to (3.47), i.e. the operator mapping  $(\tilde{v}_i)_{i=I}^\infty$  to  $(\tilde{u}_i)_{i=I}^\infty$ . When we put

$$(3.63) \quad (Gu)_i = h \sum_{j=I}^{i-1} a((i-j)h) \beta_{i,j}(h, -m) u_j ,$$

(3.44) can be rewritten in the form

$$(3.64) \quad u_i = P_I(h)(Gu + \tilde{v})_i , \quad i > I .$$

It follows from (3.61) that  $P_I(h)$  is a bounded operator. Moreover, (3.46) implies that  $G$  has small norm. Therefore,  $I - P_I(h)G$  is a nonsingular operator from  $A_{h,I}^{\sigma-\varepsilon}$  into  $B_{h,I}^{\sigma-\varepsilon}$ , and the norm of the inverse is bounded uniformly with respect to  $h$ ,  $t_{-m}$  and  $\varepsilon$ . Therefore,

$$(3.65) \quad \sup_{i>0} e^{i(\sigma-\varepsilon)h} |u_i| = \lim_{i \rightarrow \infty} |u_i| + |\lim_{i \rightarrow \infty} u_i| < c \|\hat{v}\| y_h^{\sigma-\varepsilon}$$

and, summarizing, we obtain

$$(3.66) \quad \|\hat{u}\|_{x_h^{\sigma-\varepsilon}} < \text{const} \|\hat{v}\|_{y_h^{\sigma-\varepsilon}}$$

where the constant is independent of  $h$ ,  $t_{-m}$  and  $\varepsilon \in [0, \varepsilon_0]$ , where  $\varepsilon_0 < \sigma$ . This concludes the stability proof for  $\mu = 0$ .

We briefly sketch the stability proof for  $\mu > 0$ . Equation (3.19) now takes the form

$$(3.67) \quad \mu \frac{u_i - u_{i-1}}{h} = -H_i(h, -m)u_i + h \sum_{j=-m+1}^{i-1} a((i-j)h) \left( 2 \frac{y^3(t_i)}{y^3(t_j)} + 1 \right) u_j + v_i$$

$$i > -m$$

where

$$(3.68) \quad H_i(h, -m) = 3 \int_{-\infty}^{-m} a(t_i - s) y^2(t_i) ds + 3h \sum_{j=-m+1}^{i-1} a((i-j)h) \frac{y^2(t_i)}{y^2(t_j)} - \\ - \alpha f(t_i) \cdot y^{\alpha-1}(t_i) .$$

With  $D(h)$  as in (3.46),

$$(3.69) \quad H_i(h, -m) = \frac{3}{D(h)} + \begin{cases} O(e^{-\sigma t_i}) & \text{as } t_i \rightarrow \infty \\ O(h) + O(e^{\sigma t_i}) & \text{as } t_i \rightarrow -\infty . \end{cases}$$

We can therefore use a similar perturbation approach as before, i.e., for  $t_i \rightarrow \infty$ , (3.67) is regarded as a perturbation of the problem

$$(3.70) \quad \mu \frac{\tilde{u}_i - \tilde{u}_{i-1}}{h} = - \frac{3}{D(h)} \tilde{u}_i + 3h \sum_{j=-m+1}^{i-1} a((i-j)h) \tilde{u}_j + \tilde{v}_i .$$

This can be rewritten as follows

$$(3.71) \quad \tilde{u}_i = \gamma(h, \mu) \tilde{u}_{i-1} + \delta(h, \mu) h \sum_{j=-m+1}^{i-2} a((i-j)h) \tilde{u}_j + \tilde{v}_i$$

where

$$\gamma(h, \mu) = \left( \frac{\mu}{h} + 3ha(h) \right) \left( \frac{\mu}{h} + \frac{3}{D(h)} \right)^{-1}, \quad \delta(h, \mu) = 3 \left( \frac{\mu}{h} + \frac{3}{D(h)} \right)^{-1}.$$

We substitute

$$(3.72) \quad \begin{aligned} \tilde{g}_{i-1}^{\ell} &= h \kappa_{\ell} \sum_{j=-m+1}^{i-2} e^{-\lambda_{\ell}(t_{i-1}-t_j)} \tilde{u}_j \\ \tilde{z}_{i-1} &= \sum_{\ell=1}^N e^{-\lambda_{\ell} h} \tilde{g}_{i-1}^{\ell}. \end{aligned}$$

We set  $\tilde{w}_i = (\tilde{u}_i, \tilde{z}_i, \tilde{g}_i^1, \dots, \tilde{g}_i^N)$ ,  $\tilde{\varphi}_i = (\delta(h, \mu)(\tilde{v}_i - \tilde{v}_{i-1}), 0, \dots, 0)$ . Then

(3.70) is equivalent to the system

$$(3.73) \quad \tilde{w}_i = (I + F(h, \mu)) \tilde{w}_{i-1} + \tilde{\varphi}_i$$

with

$$(3.74) \quad F(h, \mu) = \begin{pmatrix} \gamma(h, \mu) - 1 & \delta(h, \mu) & 0 & \dots & 0 \\ h \sum_{\ell=1}^N \kappa_{\ell} e^{-2\lambda_{\ell} h} & 0 & e^{-\lambda_1 h} (e^{-\lambda_1 h} - 1) & \dots & e^{-\lambda_N h} (e^{-\lambda_N h} - 1) \\ h \kappa_1 e^{-\lambda_1 h} & 0 & e^{-\lambda_1 h} - 1 & & \\ \vdots & \vdots & \ddots & \ddots & 0 \\ h \kappa_N e^{-\lambda_N h} & 0 & & & e^{-\lambda_N h} - 1 \end{pmatrix}$$

As before, it can be shown that the characteristic polynomial of  $D_h F(0, \mu)$  has the same roots as (2.3), except for the fact that 0 is a double rather than a simple root. Moreover, 0 is an exact eigenvalue of  $F(h, \mu)$ .

A proof analogous to the one for  $\mu = 0$  shows the stability for  $\mu$  fixed and sufficiently small  $h$ . For the limit  $\mu \rightarrow 0$ , a different argument is needed. When we substitute in (3.73)

$$(3.75) \quad \tilde{u}_i + \frac{\delta(h, \mu)}{\gamma(h, \mu) - 1} \tilde{z}_i = \tilde{p}_i \quad , \quad \tilde{z}_i = \tilde{q}_i$$

we obtain a system of difference equations of the form (3.73) with  $F(h, \mu)$  substituted by a matrix of the following form

$$(3.76) \quad \tilde{F}(h, \mu) = \begin{pmatrix} \gamma(h, \mu) - 1 & O(h) \\ O(h) & h \cdot \frac{d}{dh} A(0, 0) + O(h^2) \end{pmatrix}$$

Moreover, an estimate of the form  $-1 < \gamma(h, \mu) - 1 < -\omega \min(C_1, C_2) \frac{h}{\mu}$  holds where  $\omega > 0$ . Thus for small  $\mu$ ,  $|\gamma-1| \gg h$ . It is easy to conclude from this that there is a coordinate transformation close to the identity which transforms  $\tilde{F}$  to the form

$$(3.77) \quad \tilde{F} = \begin{pmatrix} \gamma(h, \mu) - 1 + O(h) & 0 \\ 0 & h \frac{d}{dh} A(0, 0) + o(h) \end{pmatrix}$$

Stability follows from the above estimate for  $\gamma - 1$  and an analysis of the eigenvalues of  $\frac{d}{dh} A(0, 0)$  given by (3.39). For  $t_i \rightarrow -\infty$  a similar argument holds, but  $D(h)$  is to be replaced by a different constant  $\tilde{D}(h, t_{-m})$ . From these considerations we see that

$$(3.78) \quad \|L_{h,m}^{-1}\|_{Y_h^{\sigma-\varepsilon} + X_h^{\sigma-\varepsilon}} \leq \text{const.}$$

with a constant independent of  $h$ ,  $m$ ,  $\mu$  and  $\varepsilon \in [0, \varepsilon_0]$ ,  $\varepsilon_0 < \sigma$ .

It is practically important to assure stability not just for  $h$  sufficiently small, but also for arbitrary  $h$ . Recall that linearizing  $e^{-\lambda_i h} - 1$  with respect to  $h$  is only justified if  $h \ll \frac{1}{\lambda_i}$ . The matrix  $F(h, \mu)$  for arbitrary  $h$  has the same form as for  $h$  small, if the following substitutions are made:  $\lambda_i \rightarrow \frac{e^{-\lambda_i h} - 1}{h}$ ,  $K_i \rightarrow K_i e^{-2\lambda_i h}$ ,  $\mu \rightarrow \mu + \frac{3h}{D(h)}$ . If  $f$  has compact support, this is sufficient to ensure stability. If the support of  $f$  is not compact, stability for arbitrary  $h$  can be assured if the following modification is made: In (3.3) the integral

$\int_{-\infty}^{t_i - m} a(t_i - s) ds$  is replaced by  $h \cdot \sum_{j=-\infty}^{-m} a(t_i - t_j)$ . With this modification

the term  $O(h)$  in (3.69) vanishes, and thus the matrix for the linearized problem is asymptotically equal to  $F(h, \mu)$  both for  $t_i \rightarrow \infty$  and  $t_i \rightarrow -\infty$ . In order to apply Keller's [3] nonlinear stability concept, it is further necessary to show that the Fréchet derivatives  $D_2 F_{h,m}$  are uniformly Lipschitz continuous in a sphere

$$S_K = \{\hat{z} \in X_h^{\sigma-\varepsilon} \mid \|\hat{z} - (y(t_i) - 1)_{i=-\infty}^{\infty}\| < K\} .$$

This follows from a fairly trivial calculation, which we do not present here.

Using the fact that the global error  $(y(t_i) - y_i)_{i=-\infty}^{\infty}$  is estimated by a constant times the bound for the local error (3.12), we obtain the following theorem:

#### Theorem 3.1

The discretization scheme (3.3), (3.4) has a unique solution for all  $f \in X_h^{\sigma-1}$ , where  $\sigma$  is as of Theorem 2.3, this solution  $\hat{y} = (y_i)_{i=-\infty}^{\infty}$  can be calculated by the Newton procedure which is second order convergent from a

sphere of starting values which does not shrink to  $\emptyset$  as  $h \rightarrow 0$ ,  $t_{-m} \rightarrow -\infty$ ,  $i \rightarrow 0$  and the convergence estimate

$$(3.79) \quad \|(y_i - y(t_i))_{i=-\infty}^{\infty}\|_{x_h^{\sigma-\varepsilon}} \leq \text{const.} (h + o(e^{-\varepsilon|t_{-m}|}))$$

holds for  $h$  sufficiently small and  $|t_{-m}|$  sufficiently large. The constant is independent of  $h$ ,  $t_{-m}$ ,  $\mu \in [0, \infty]$ ,  $\varepsilon \in [0, \varepsilon_0]$ ,  $\varepsilon_0 < \sigma$ .

This implies that the Newton procedure for the solution of (3.6) can be safely applied, that the  $(y_i)_{i=-\infty}^{\infty}$  do not exhibit boundary layer-like behavior and that

$$(3.80) \quad |y_i - y(t_i)| \leq \text{const.} e^{(\sigma-\varepsilon)t_i} (h + o(e^{-\varepsilon|t_{-m}|})), \quad t_i < 0$$

$$(3.81) \quad |\lim_{i \rightarrow \infty} y_i - y(\infty)| \leq \text{const.} (h + o(e^{-\varepsilon|t_{-m}|}))$$

$$(3.82) \quad |(y_i - \lim_{i \rightarrow \infty} y_i) - (y(t_i) - y(\infty))| \leq \text{const.} e^{-(\sigma-\varepsilon)t_i} (h + o(e^{-\varepsilon|t_{-m}|})), \quad t_i > 0.$$

and the order of convergence is independent of  $\mu \in [0, \infty]$ .

Obviously if  $f$  is only supported on  $[T_1, T_2]$ , then the term  $o(e^{-\varepsilon|t_{-m}|})$  disappears from the error estimate if  $t_{-m} < T_1$ .

The discretization we used was derived from the integral equation. In chapter 2 we transformed to a system of ordinary differential equations. In fact, up to terms of order  $O(h)$ , our discretization method corresponds to a discretization scheme for the ODE system (2.1). Namely, if we put

$$q_{\ell, i} = h \sum_{j=-m+1}^i K_{\ell} e^{-\lambda_{\ell}(t_i - t_j)} \frac{1}{y_j^2}, \quad q_{\ell, -m} = 0$$

$$h_{\ell,i} = h \sum_{j=-m+1}^i K_\ell e^{-\lambda_\ell(t_i - t_j)} y_j \quad , \quad h_{\ell,-m} = 0$$

our discretized equation reads as follows:

$$\begin{aligned}
 (a) \quad & -\mu \left( \frac{y_i - y_{i-1}}{h} \right) = \sum_{\ell=1}^N e^{-\lambda_\ell h} (g_{\ell,i-1} y_i^3 - h_{\ell,i-1}) \\
 (b) \quad & \frac{g_{\ell,i} - g_{\ell,i-1}}{h} = \frac{K_\ell}{2} + \frac{(e^{-\lambda_\ell h} - 1)}{h} g_{\ell,i-1} \\
 (c) \quad & \frac{h_{\ell,i} - h_{\ell,i-1}}{h} = K_\ell y_i + \frac{(e^{-\lambda_\ell h} - 1)}{h} h_{\ell,i-1} .
 \end{aligned}
 \tag{3.83}$$

By calculating  $g_{\ell,i}$ ,  $h_{\ell,i}$  for  $\ell = 1(1)N$  from (3.83)(b), (3.83)(c) and by inserting these quantities into (3.83)(a) an equation of the form (3.6) is obtained (in each time step). Theorem 3.1 now implies that the root of this equation can be safely obtained by the Newton procedure which is second order accurate from a sphere of starting values whose radius is independent of  $h$ ,  $t_{-m}$ ,  $\mu \in [0, \infty)$  and  $i > -m$ .

This provides us with a very efficient method to solve the approximating problems and Theorem 3.1 makes sure that the qualitative properties of the solution of (1.1) carry over to the approximate solutions.

The exponential decay of the solution encourages one to attempt using variable mesh sizes of the form

$$\hat{h}_i = h e^{-\sigma |t_i|} .
 \tag{3.84}$$

It can be expected that convergence of the order one in  $\hat{h}$ , i.e. the estimate would be  $O(\hat{h}) + o(e^{-\sigma|\hat{t}_m|})$ , would follow in  $\ell^\infty$  but the exponential decay property of the approximate solution would be lost. In the case of boundary value problems for ordinary differential equations on infinite intervals this has been shown in [7].

A further problem that should be mentioned is which higher order discretization schemes could be employed. It is fairly clear from our analysis that polynomial collocation methods using Radau points (see [12]) could be used and the exponential decay property and the uniform convergence as  $\mu \rightarrow 0$  would be recovered.

4 Numerical Results

For the computations we used the kernel  $a(u) = \sum_{i=1}^8 K_i e^{-\lambda_i u}$  with the following constants  $K_i$  and  $\lambda_i$

i	$\lambda_i (\text{sec}^{-1})$	$K_i (\text{Nm}^{-2} \text{sec}^{-1})$
1	$10^{-3}$	$1 \times 10^{-3}$
2	$10^{-2}$	$1.8 \times 10^0$
3	$10^{-1}$	$1.89 \times 10^2$
4	1	$9.8 \times 10^3$
5	10	$2.67 \times 10^5$
6	$10^2$	$5.86 \times 10^6$
7	$10^3$	$9.48 \times 10^7$
8	$10^4$	$1.29 \times 10^9$

These numbers were obtained by Laun [4] from an experimental fit for a polyethylene melt at 150°C, which he calls "Melt 1".

The parameter  $\mu$  is physically identified as three times the Newtonian contribution to the viscosity. Experimental values are not available, and theoretically  $\mu$  is either a solvent viscosity (for polymer solutions) or it results from fractions of low molecular weight (for melts). The value of  $\mu$  has to be compared to the viscosity resulting from the memory, which, for constant shear rate, is given by  $\sum_{i=1}^8 K_i \lambda_i^{-2} \approx 50000 \text{ Nm}^{-2} \text{sec}$ . One would expect  $\mu$  to influence the solution significantly only if it exceeds this value.

This is verified by our computations. In the plots the scale for  $y$  is on the left, the scale for  $f$  is on the right.  $y$  is measured in multiples of the length (for the filament,  $\alpha = 2$ ), or, respectively, the thickness (for the sheet,  $\alpha = \frac{1}{2}$ ) at  $t = -\infty$ ;  $f$  denotes the force acting on the ends of the filament or the edges of the sheet divided by the cross-sectional area in the undeformed state at  $t = -\infty$ ,  $f$  is expressed in  $\frac{N}{m^2}$ . The time is measured in seconds.  $f$  is always plotted by dashed lines,  $y$  by full lines.

All plots except figures 7, 8 were made for  $\alpha = 2$ , the case of the filament. In figures 1 - 11 (except 6), the force  $f$  is of the form

$$f(t) = \begin{cases} 0 & |t| > a_2 \\ f_{\max} \exp\left\{a_0^2 - \frac{a_1^2}{a_2^2 - t^2}\right\} & |t| \leq a_2 \end{cases}$$

with  $a_0^2 - \frac{a_1^2}{a_2^2} = 0$ . Such an  $f$  is in  $C^\infty(\mathbb{R}, \mathbb{R})$  and has the compact support  $[-a_2, a_2]$ .

The parameter  $\mu$  is zero in figures 1 - 9. In figures 1 - 5 we have chosen various values of  $f_{\max}$ ,  $a_0$  and  $a_1$ , as can be seen from the diagrams. The calculations were done for larger time intervals than the plots, thus yielding approximations for  $y(\infty)$ . For figures 1 - 5, the approximate values of  $y(\infty)$  are as follows:

fig.	1	2	3	4	5*
$y(\infty)$	1.07	1.15	1.11	1.26	1.13

\*(in this figure supp  $f$  is different from the previous ones)

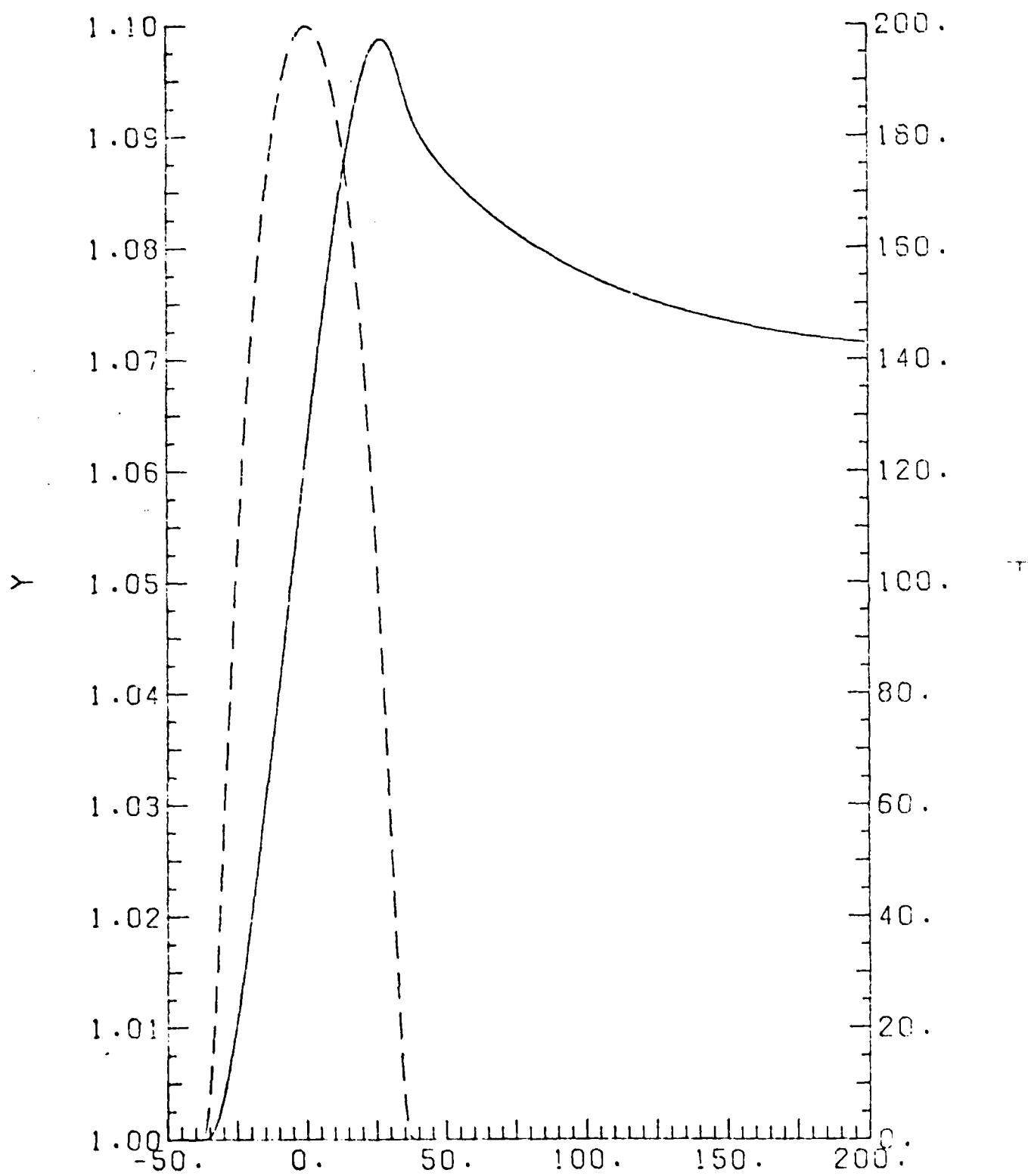


Figure 1

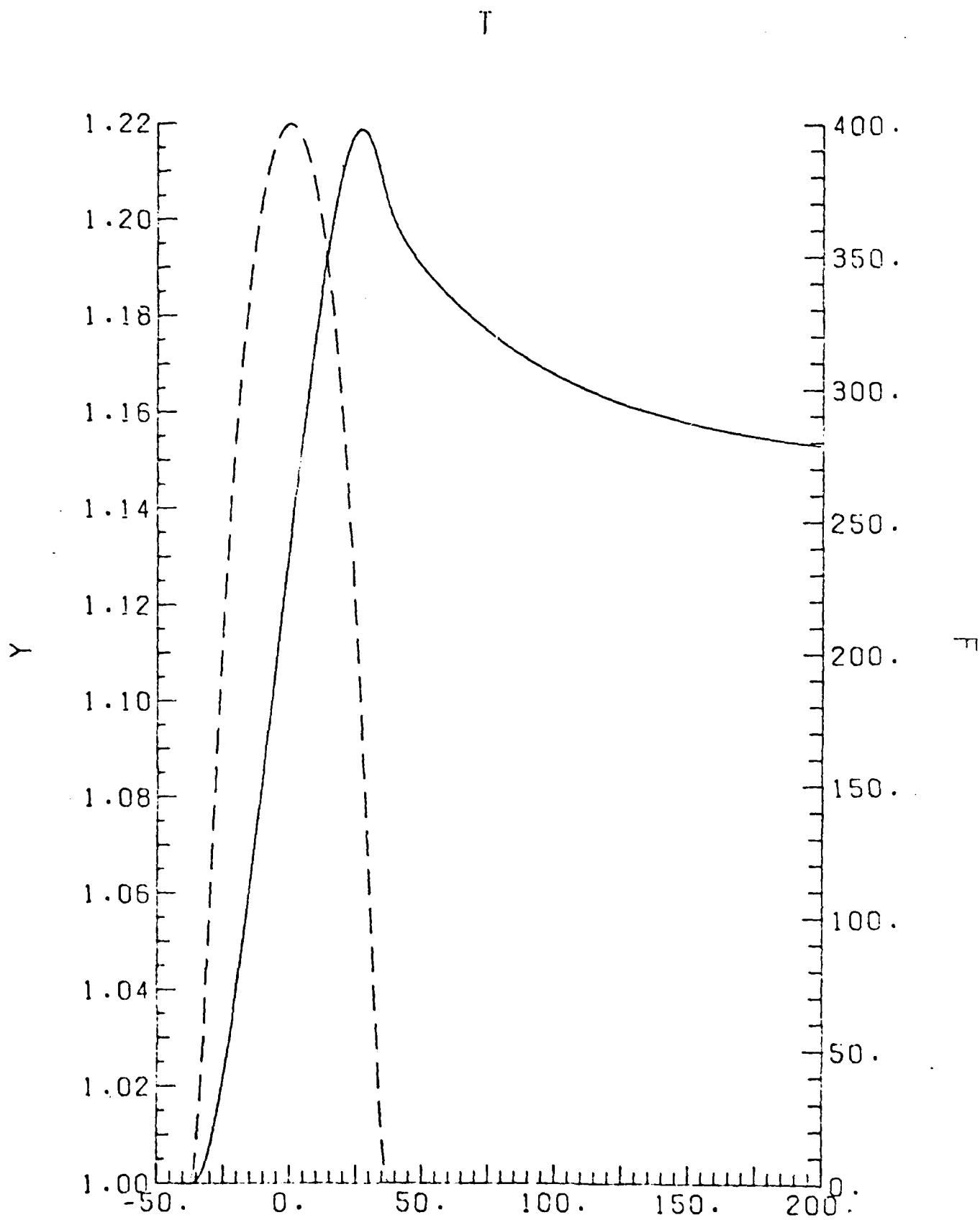


Figure 2

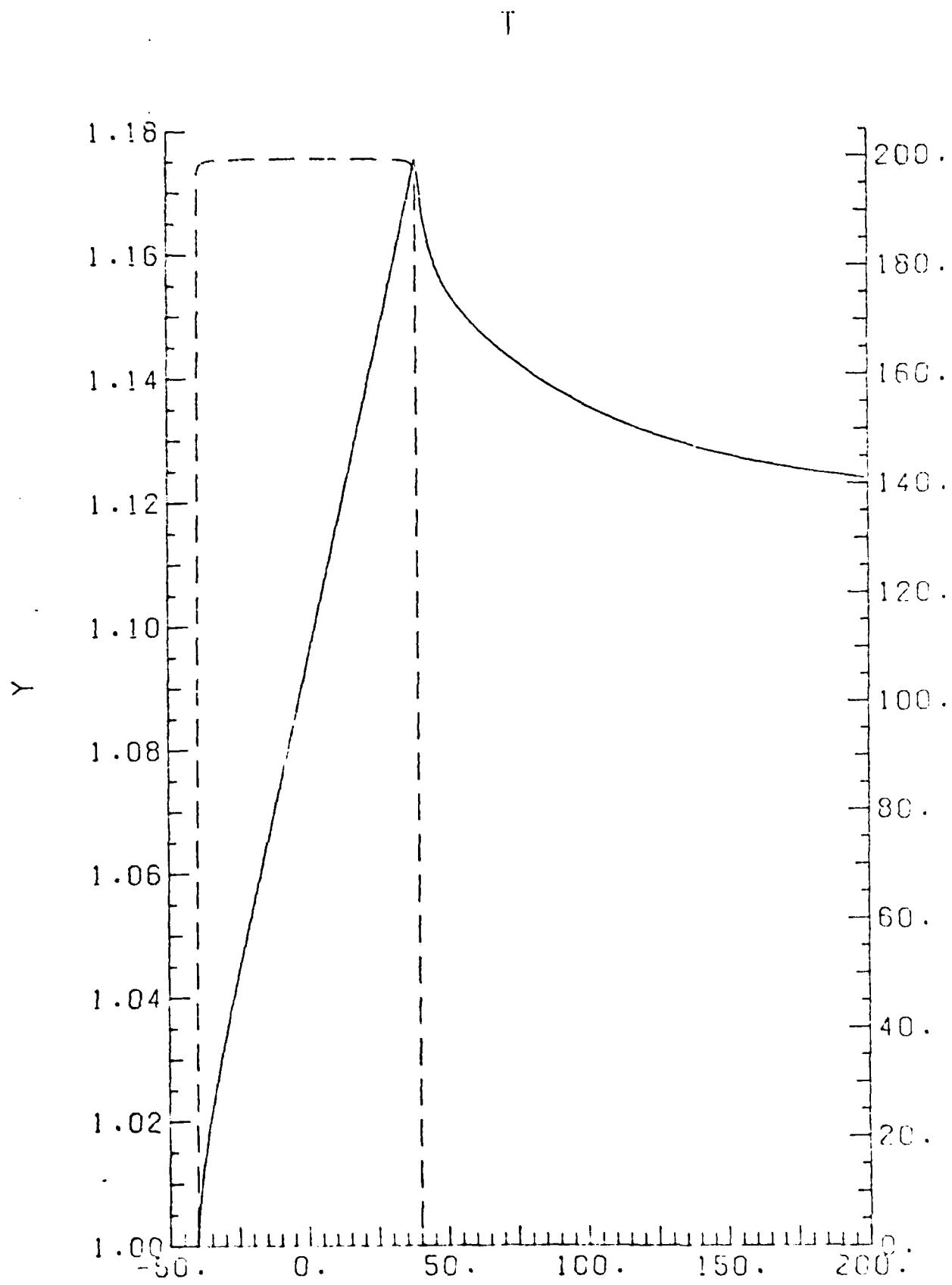


Figure 3

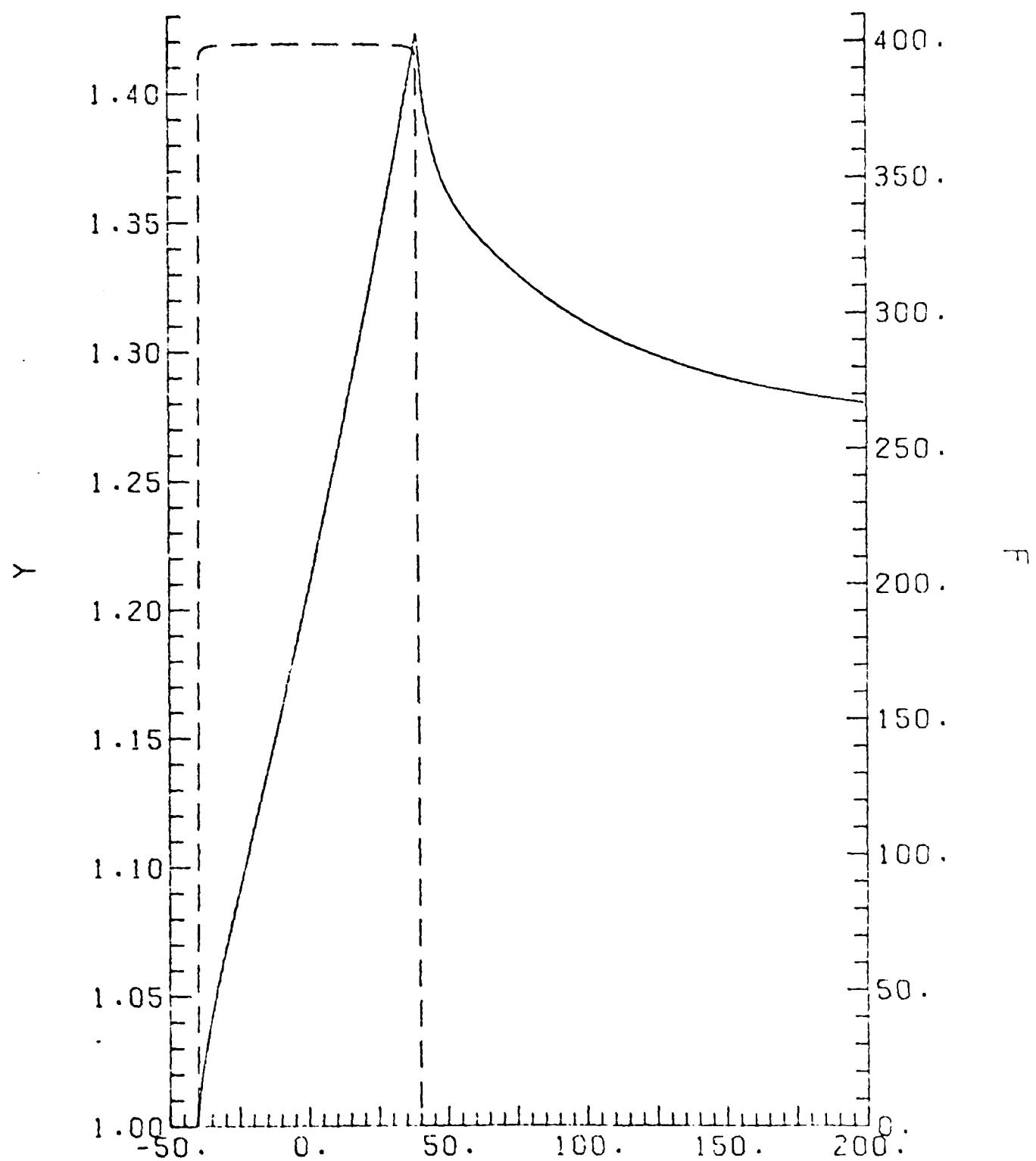


Figure 4

T

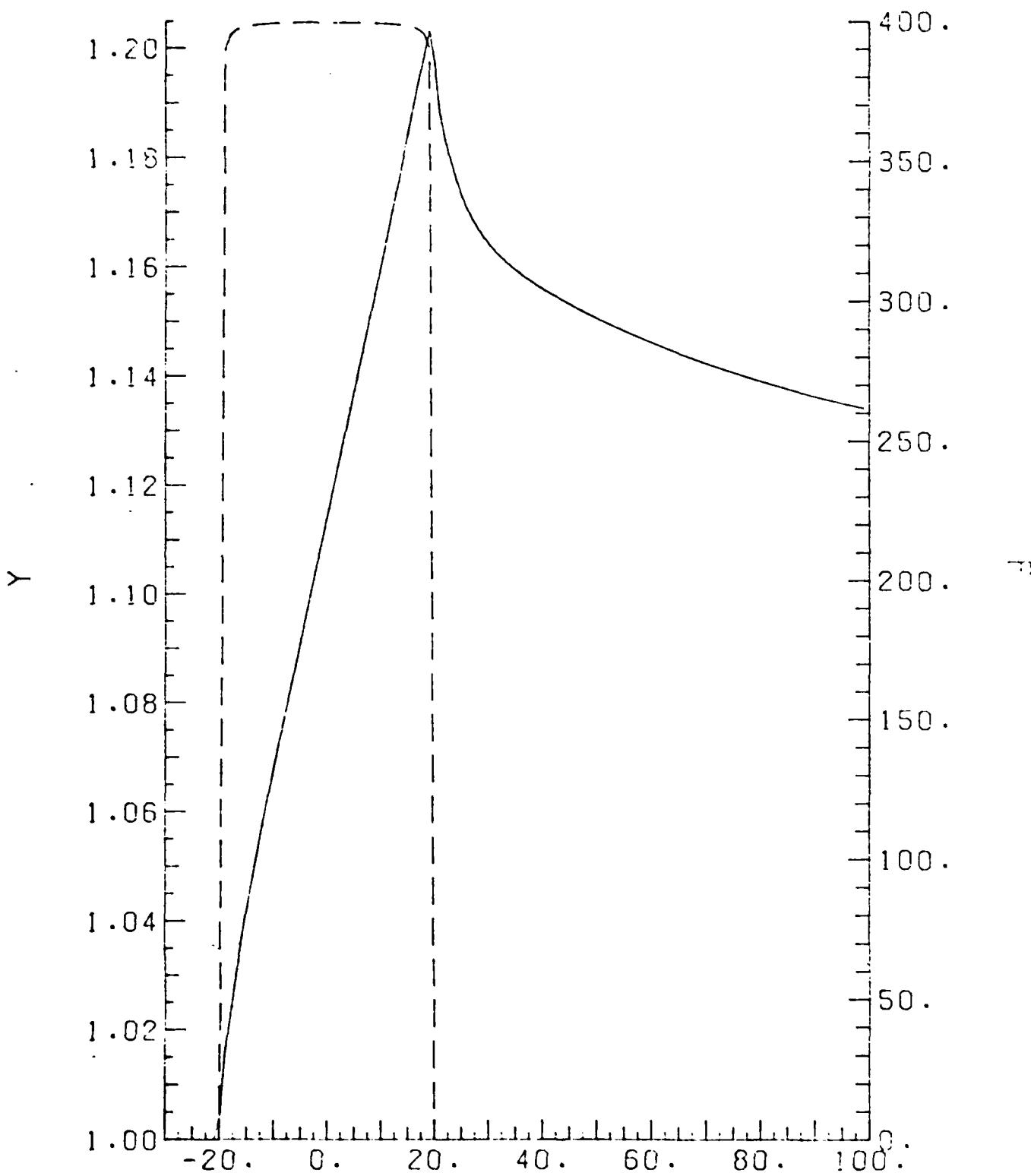


Figure 5

T

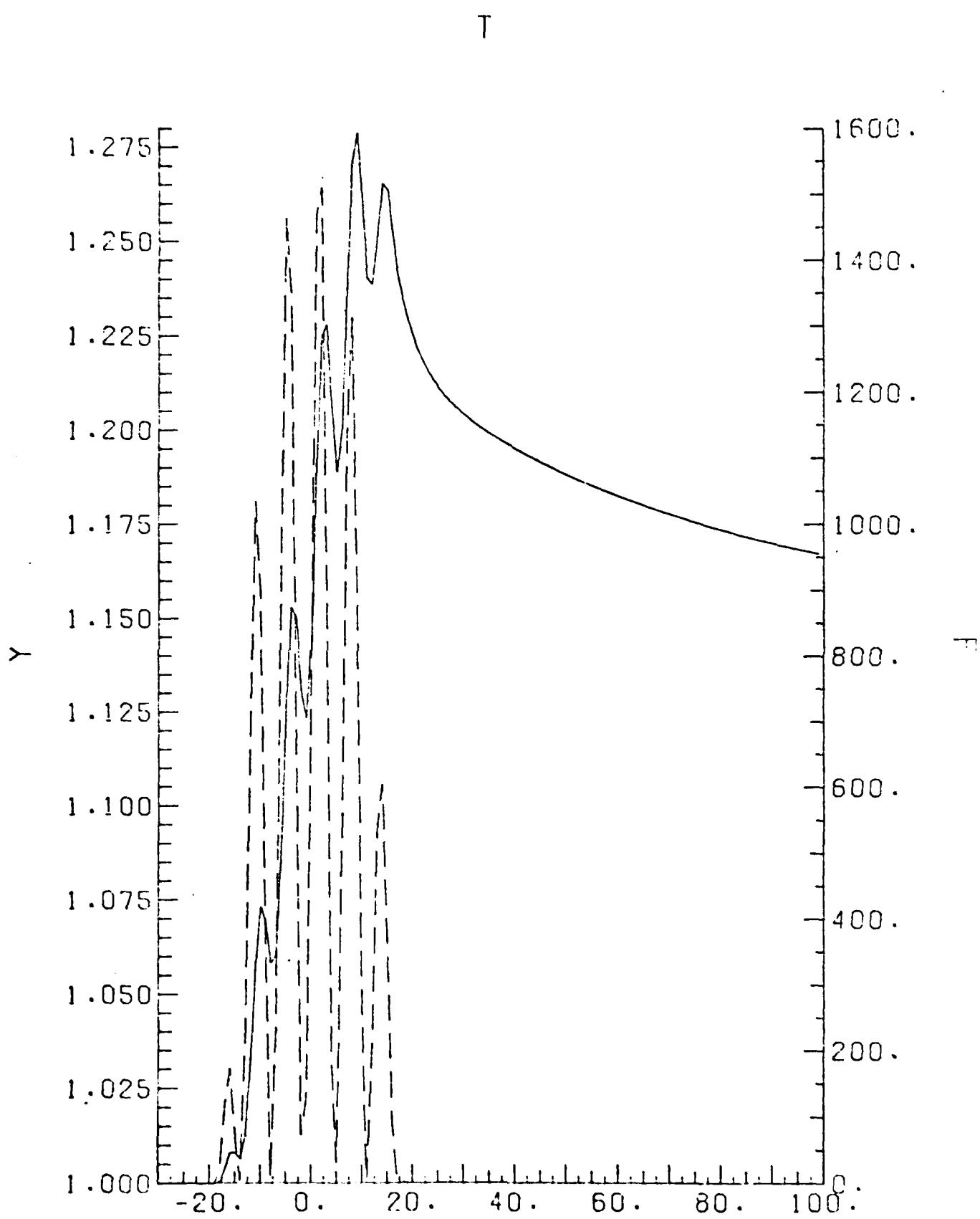


Figure 6

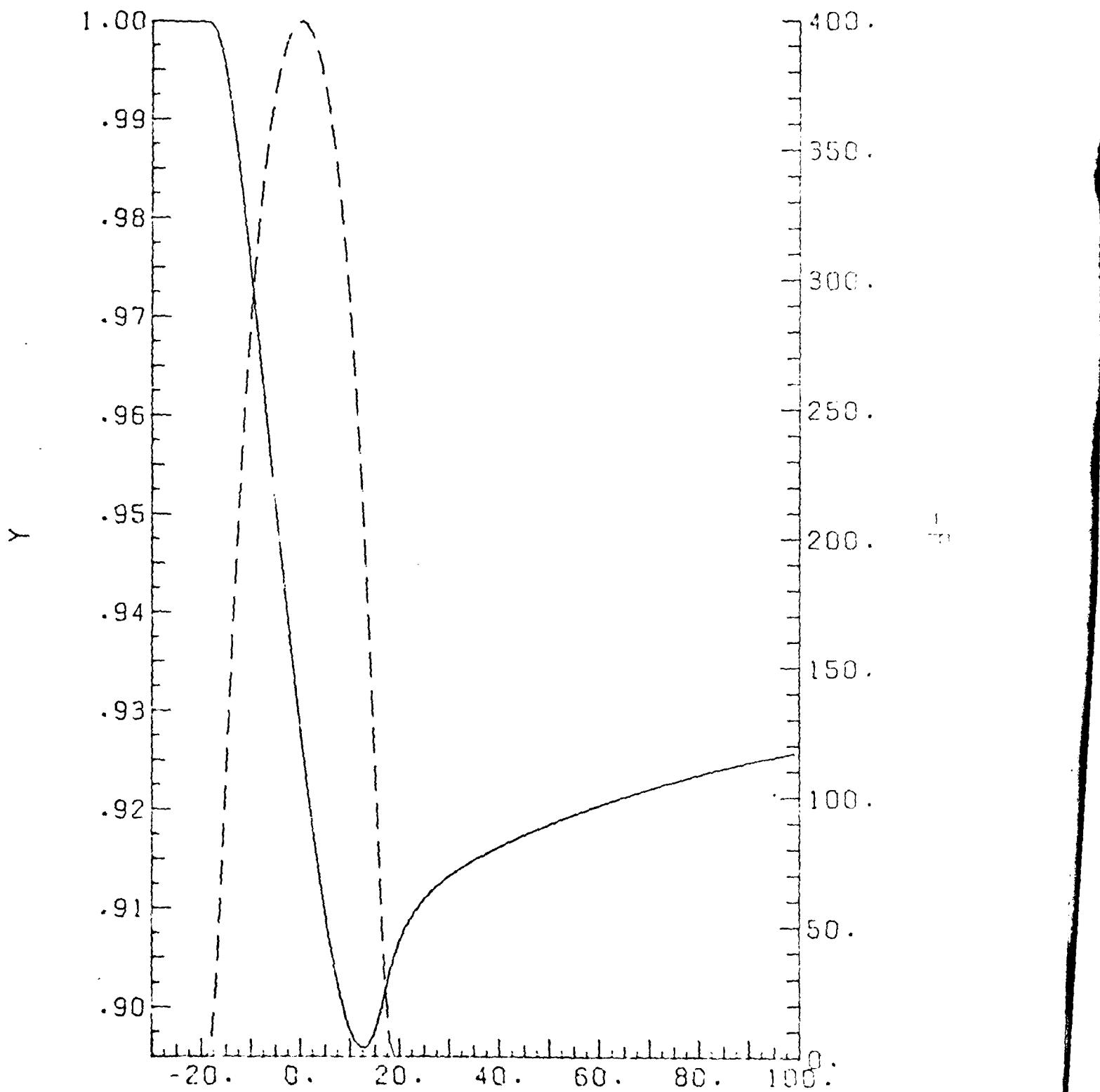


Figure 7

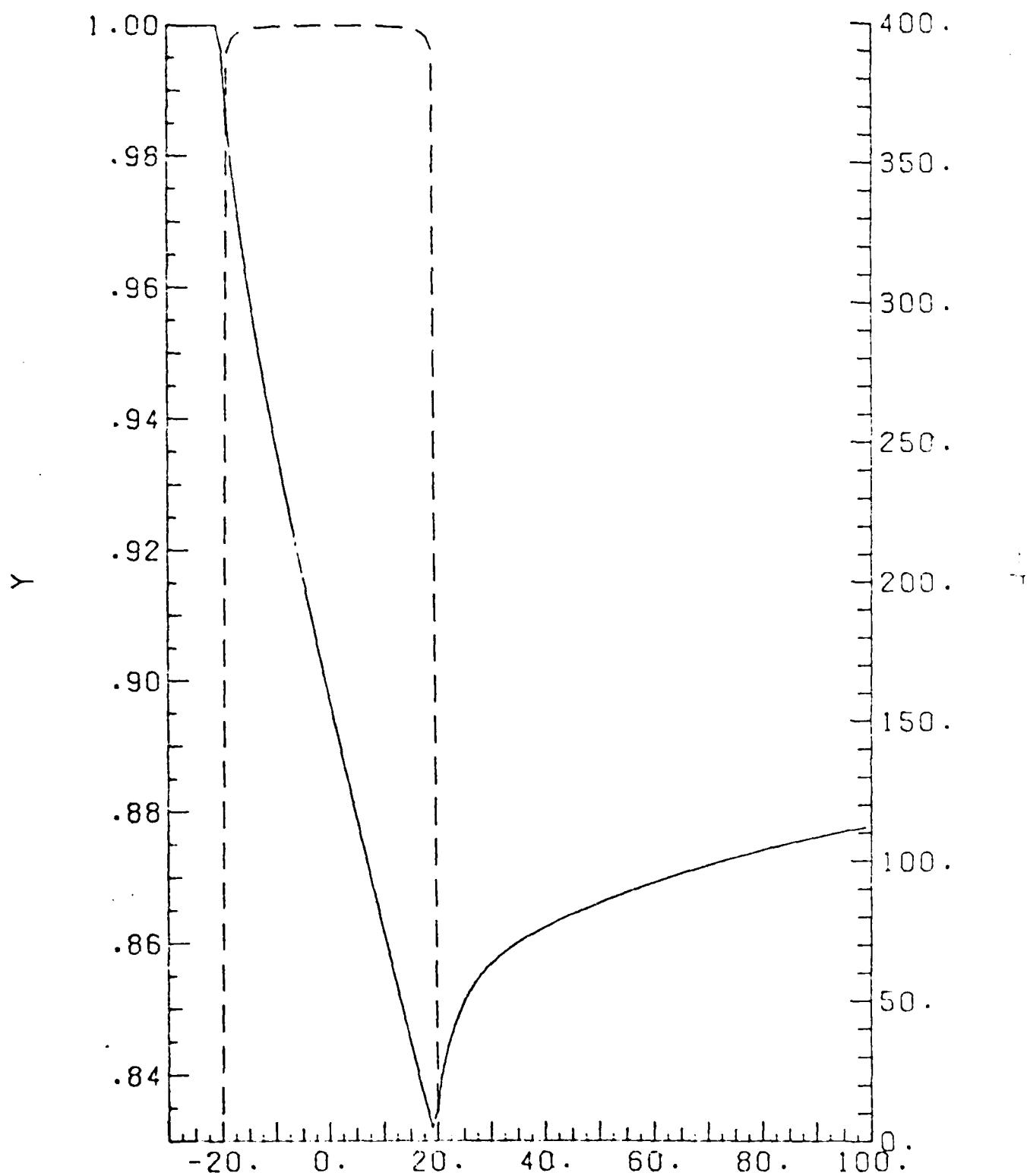


Figure 8

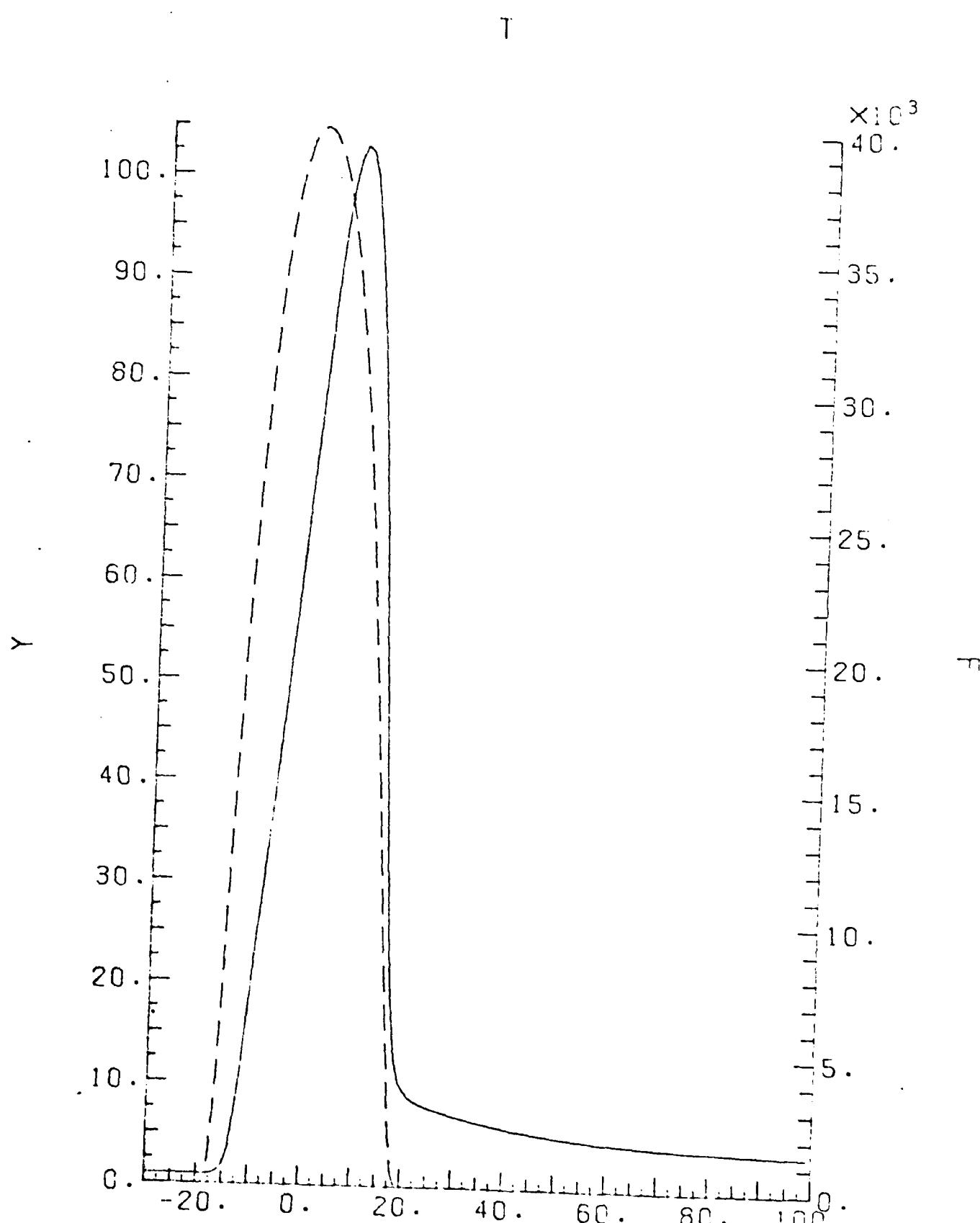


Figure 9

$\mu = 0$

T

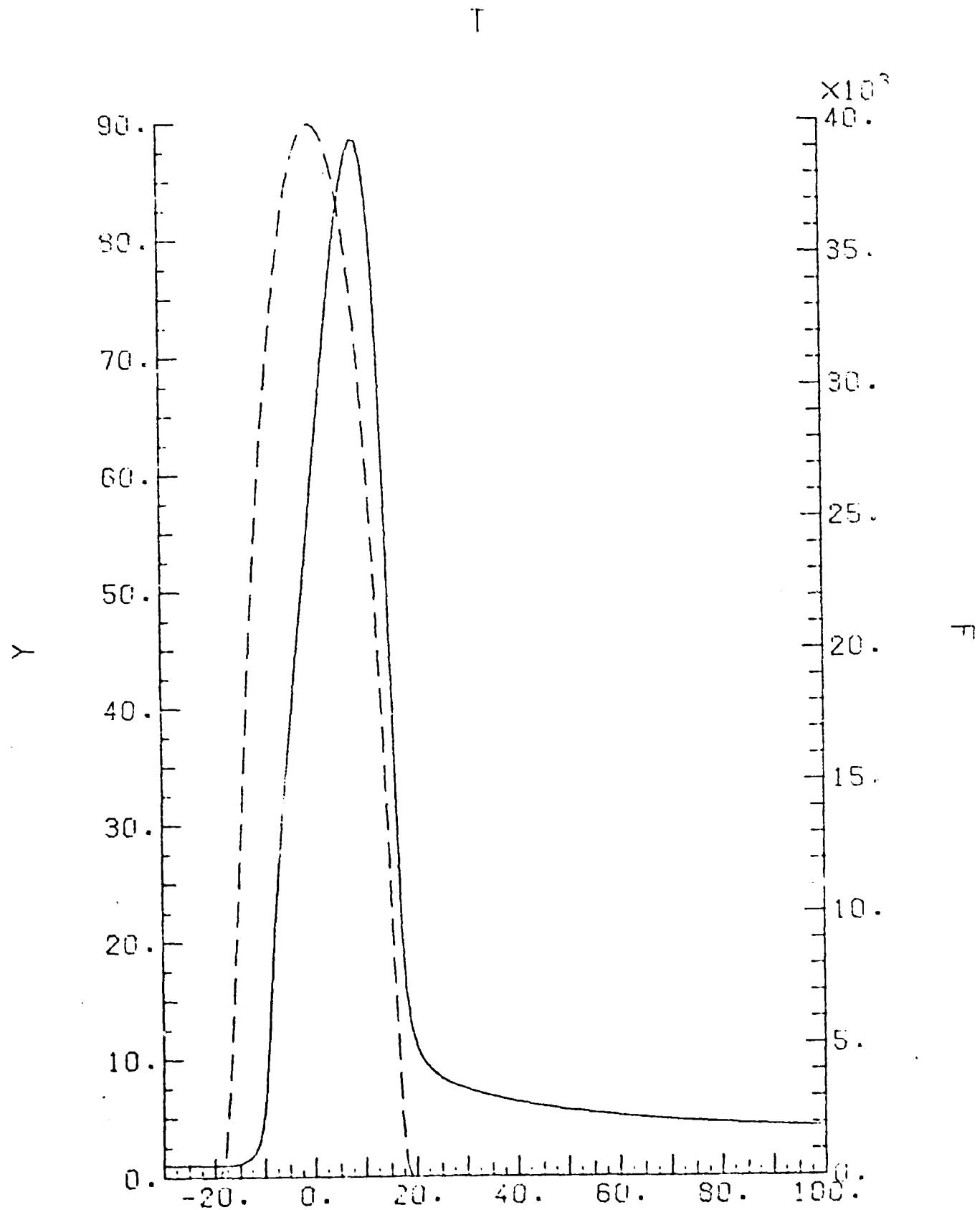


Figure 10

$$\mu = 10^5$$

T

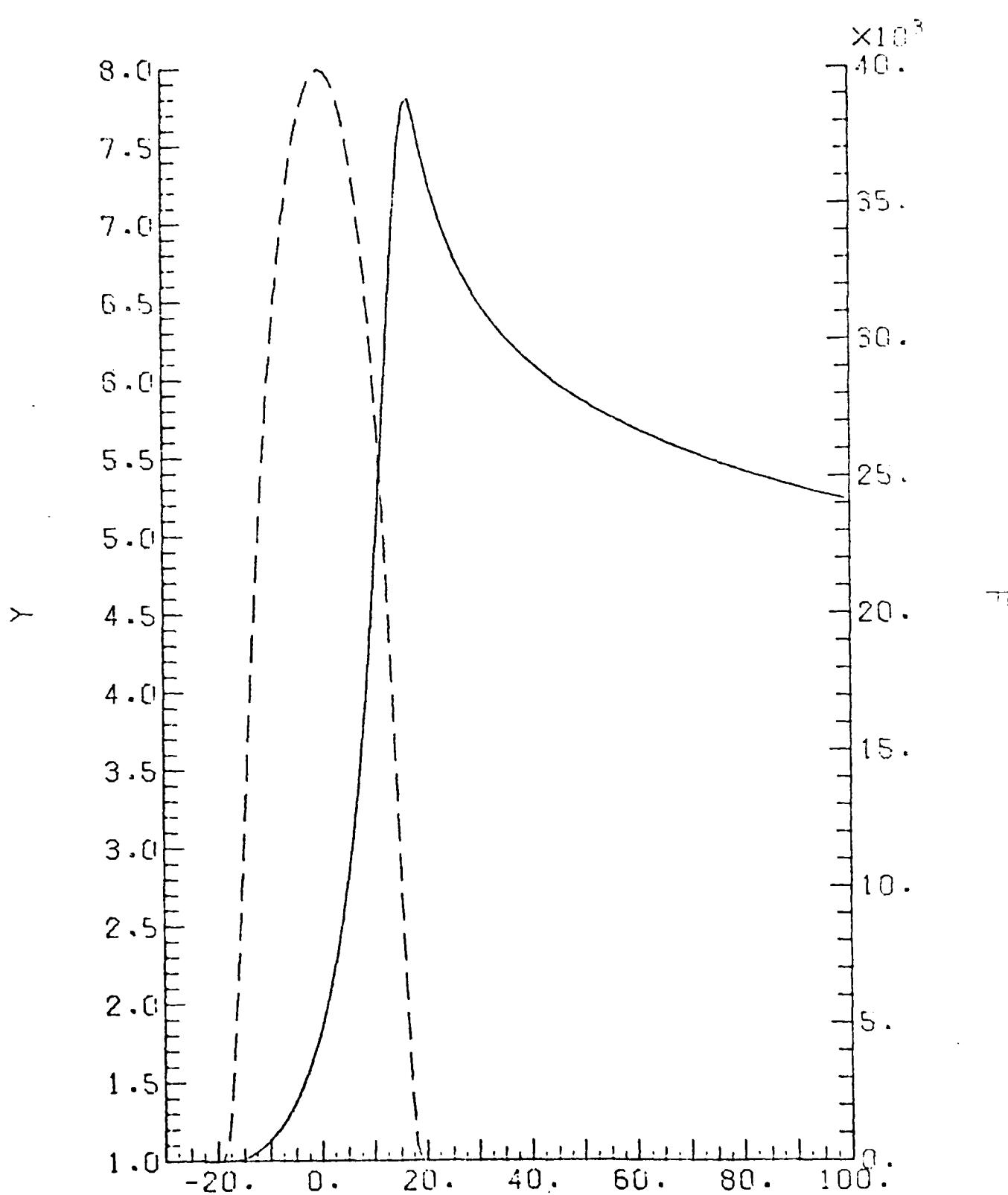


Figure 11

$$\mu = 10^6$$

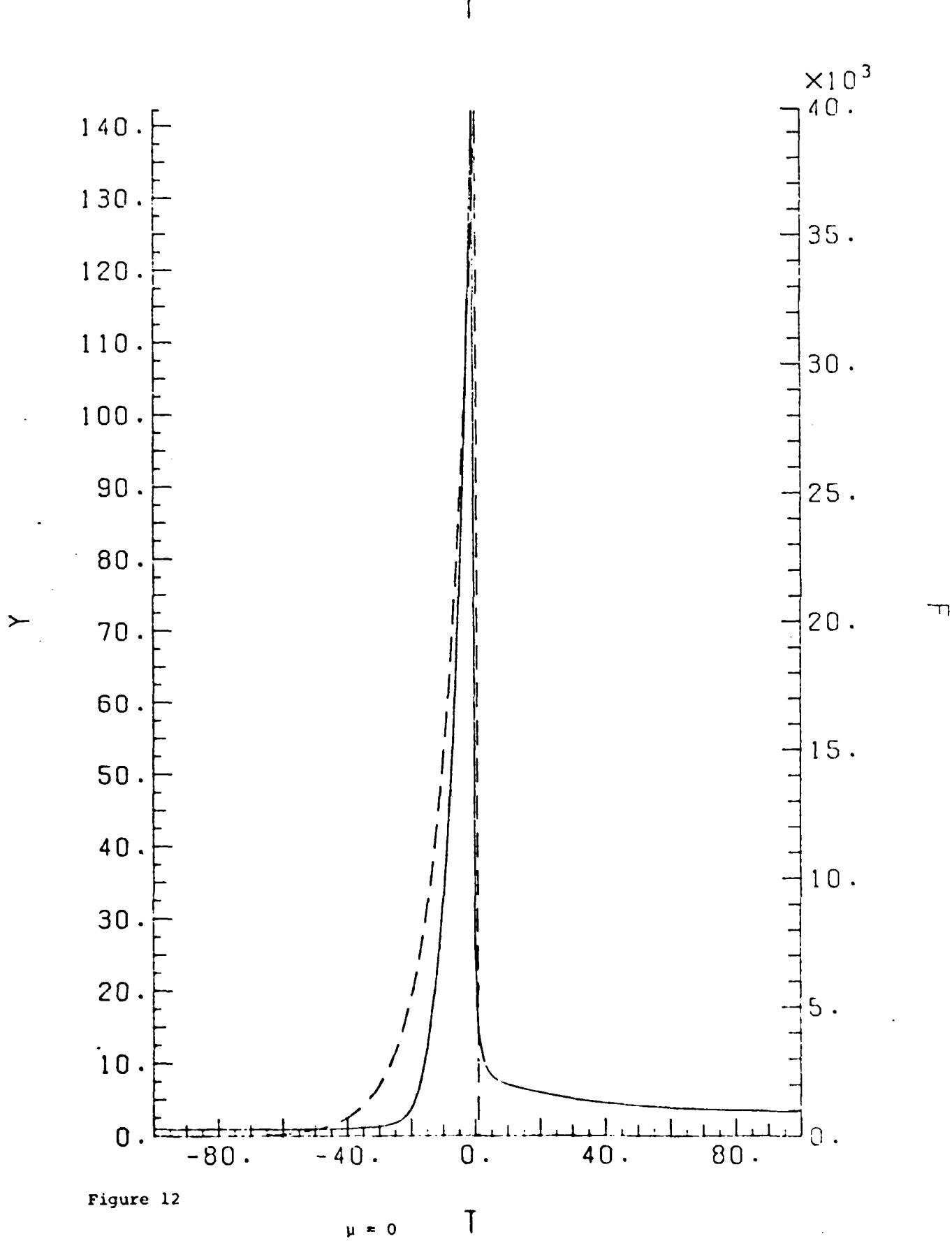


Figure 12

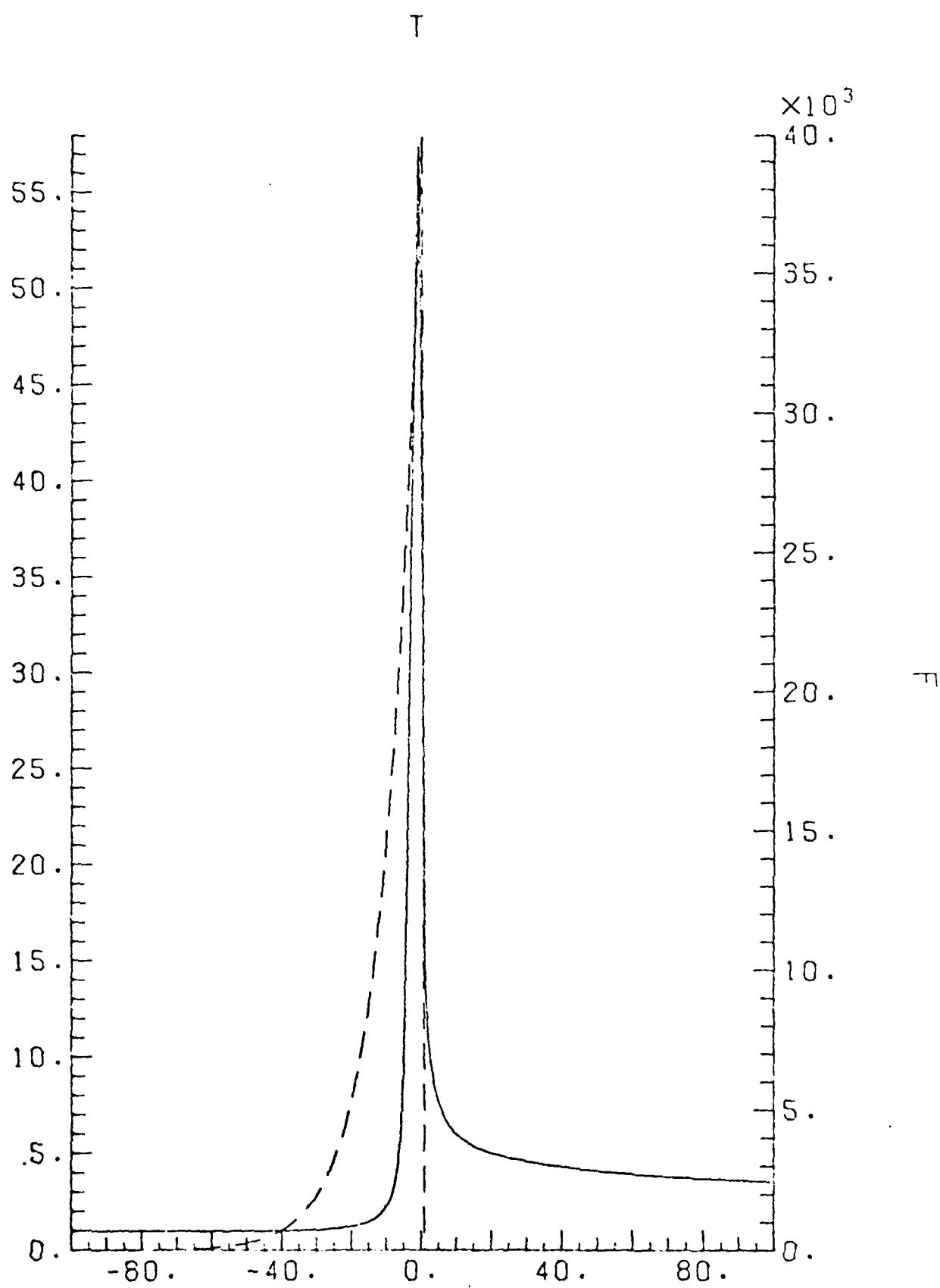


Figure 13

$$\mu = 20 \times 10^4$$

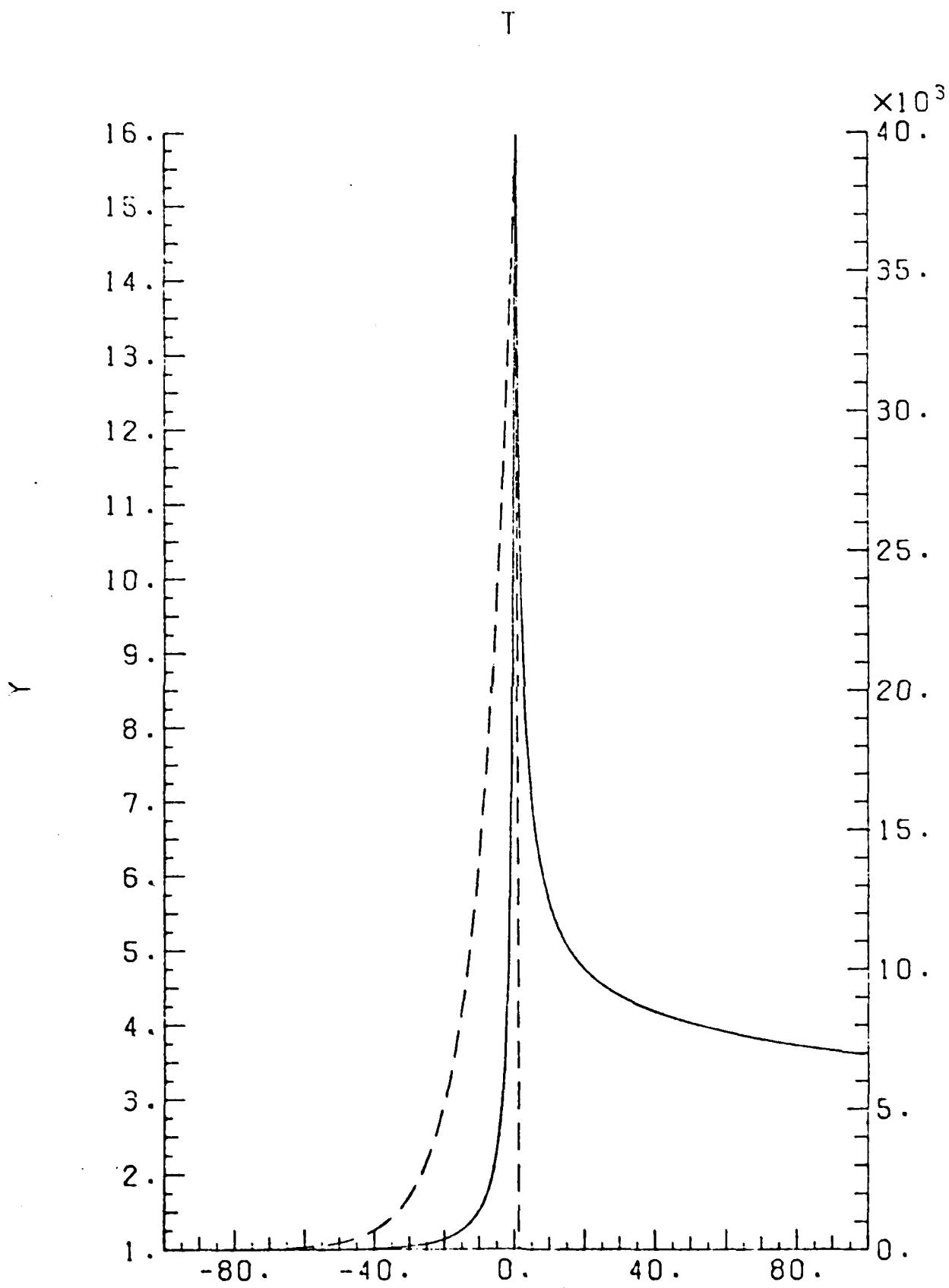


Figure 14

$$\mu = 35 \times 10^4$$

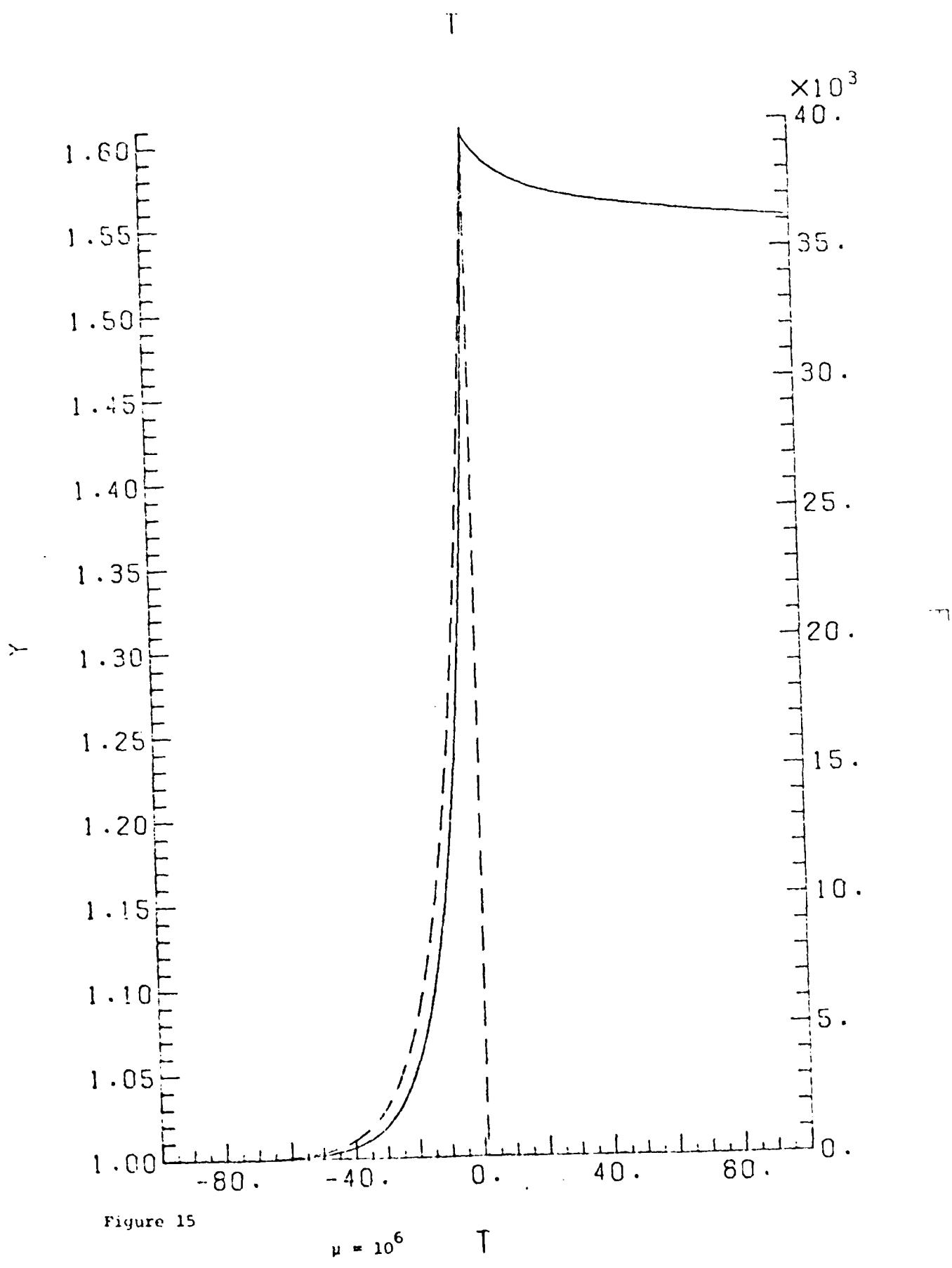


Figure 15

These numbers indicate that, roughly speaking, the value of  $y(\infty) - 1$  is proportional to  $\int_{-\infty}^{\infty} f(t)dt$ . This would in fact be exact for the linearized equation.

In figure 6 an oscillating force was chosen. It is observed that the solution  $y$  "follows" the oscillations with a certain delay.

The figures 7 and 8 illustrate the case of the sheet ( $\alpha = \frac{1}{2}$ ). Here  $-f$  is plotted rather than  $f$ . The results are qualitatively similar to those in figures 1 - 5, but now we have  $y < 1$  instead of  $y > 1$ .

In figures 9 - 15, we have again  $\alpha = 2$ . In figures 9 - 11, we have chosen the same  $f$  ( $f_{\max} = 40000$ ,  $a_0 = 1$ ,  $a_1 = a_2 = 20$ ) and computed solutions for different values of  $\mu$ .

$\mu$	$y_{\max}$	$y(\infty)$
0	103	3.3
$10^5$	89	3.4
$10^6$	7.7	4.5

For  $\mu < 10000$ , no significant change was observed. For larger  $\mu$ , the effect on the maximal elongation seems to be more pronounced than the effect on the final length. Recalling the fact that  $\mu = 10000$  would correspond to a viscosity  $3 \times 10^6$  as large as that of water, it seems conceivable that for fluids like "Melt 1"  $\mu$  can be neglected.

The numbers for  $y(\infty)$  are interesting in comparison with the results of Lodge, McLeod and Nohel [6]. They showed that  $y(\infty)$  increases with  $\mu$ , if the history of  $y$  for  $t < 0$  is kept fixed. Our numbers show the same

tendency, eventually, however,  $y(\infty)$  has to decrease, since for  $\mu = \infty$  we have  $y = \text{const.}$  and thus  $y(\infty) = 1$ . We see from this that, for fixed  $f$ ,  $y(\infty)$  is not a monotone function of  $\mu$ .

For figures 12 - 15, a discontinuous force given by

$$f(t) = \begin{cases} 0 & t > 0 \\ 40000 \exp\left(\frac{t}{10}\right) & t < 0 \end{cases}$$

was used. Since in this case the filament recovers freely for  $t > 0$ , we are studying the same situation as Lodge, McLeod and Nohel [6], but we prescribe the force rather than the history of  $y$  for  $t < 0$ . By considering the intervals  $t < 0$  and  $t > 0$  separately, we can easily modify the existence and convergence theory of the previous chapters for the present case.

However, the solution does not depend continuously on  $\mu$  in the  $L^\infty$ -norm as  $\mu \rightarrow 0$ . This is because for  $\mu = 0$  the solution is discontinuous at  $t = 0$ . The following table illustrates the dependence of the maximal elongation on  $\mu$ .

$\mu$	$y_{\max}$
0	140
$2 \times 10^5$	58
$3.5 \times 10^5$	16
$10^6$	1.6

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ABSTRACT (continued)

solution which approaches a given limit as  $t \rightarrow -\infty$ ; moreover, the solution also has a limit as  $t \rightarrow +\infty$ . A numerical scheme is analyzed and convergence uniformly in  $t$  is established. Particular attention is paid to the dependence of solutions on a parameter  $\mu$ , which corresponds to a Newtonian contribution to the viscosity. It is proved that solutions converge uniformly in  $t$  as  $\mu \rightarrow 0$ , and that the convergence of the numerical scheme is also uniform in  $\mu$ .

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